ELEMENTS

OF

PRACTICAL MECHANICS,

BY

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TO WHICH IS ADDED

A Treatise

UPON

THE PRINCIPLE OF VIRTUAL VELOCITIES,

AND ITS USES IN MECHANICS.

TRANSLATED FROM THE ITALIAN,

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ADVERTISEMENT.

The favourable reception of the former part of this Work, by the scientific portion of the English public, in general, and by the University of Cambridge, in particular, has led to the translation of the remainder: and the same peculiarities, which have served to recommend that former part of the Treatise, will, it is presumed, be found to characterize this latter also, which contains an exposition of the principles of Practical Mechanics. Much, indeed, of what is here published, cannot but be familiar to the minds of those, who are already conversant with mathematical investigations; but yet much of it will probably be new to the English student, not only in form, but in substance: and, independently of all other considerations, the subject matter, now offered to the reader, must of itself be, in the highest degree, interesting to him. For the enquiries, which it involves, are of the utmost importance to the arts of civilized life.

In the first of the following Books, the author has endeavoured to deduce, from the most accurate experiments, of which the results have been pub-

lished, the most certain and extended notions, upon the measure of natural forces, and upon their mode of action. Now these forces are of two kinds. Some of them tend to produce motion; whilst others operate only to destroy, or to diminish the motion, produced by the former kind. Following, therefore, this order, he first treats of *active forces*, and then of *passive forces*, which, in other terms, are called *resistances*.

In the next Book, he considers the Equilibrium of Structures, and solves the principal problems, which belong to that branch of his subject.

In the last Book he treats of Machines, first, in a state of equilibrium; next, in a state bordering upon that of motion; and, lastly, in a state of actual motion. The advantage of considering these three states separately, is very evident; and Venturoli has thus judiciously set an example, which subsequent writers upon Mechanics will do well to imitate.

By the division, which the author has made of his materials, the difficulty of applying pure mathematics to purposes of practical utility, with any near approach to exactness of reasoning, becomes very evident. And the attempt to make this application is here so conducted, as to urge the mind of the reader into a train of thought, and to guide him to a course of experiments, which if he pursue, he may probably be enabled to accelerate the progress of

Practical Mechanics toward that desired limit of perfection, which is yet far from having been attained.

As the following Elements are a continuation of those, by the same author, the publication of which, in an English version, has preceded them, the numbering of the Books, and of the Articles, has been thence carried on, instead of having been made to begin anew. Thus, the numerical references are all of them in one series, throughout the whole Treatise.

The Appendix, upon the Principle of Virtual Velocities, properly belongs to the former part of this Work, and is, indeed, a most valuable addition to it.

Tables are inserted, at the end of the volume, by means of which, the results of experiments and calculations, expressed according to the denominations of various foreign weights and measures, may be reduced to those of the English system; and *vice versâ*. And the particular numbers are also exhibited, together with their logarithms, which are of the most frequent occurrence in mechanical computations.

Trinity College,
Feb. 10, 1823.

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ERRATA.

Page 17. l. 5 *for* acriform *read* aeriform.
34. l. 4 *for* meters *read* metres.
54. l. 17 from the bottom, *for* nearer *read* near.
93. l. 3 from the bottom, for $M a \mu$ read $M a \mu'$.
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120. l. 1 *for* Chap. VI. *read* Chap. IV.
142. l. 11 from the bottom, for $Q.m$ read $Q.\cos.m$.
144. l. 1 for $f.P.R$ read $f.P.r$.
— l. 8 *for* friction *read* frictions.
158. l. 6 from the bottom, for $n du$ read $u du$.
173. l. 1 *for* summoning *read* summing.
186. l. 14 *for* if *read* of.
190. l. 2 for dt read dh.
200. l. 5 *for* adopt *read* adapt.
204. l. 14 *before* manner *insert* same.
205. l. 1 for Px read Pz.
232. l. 6 from the bottom, *for* 1.14 *read* 1.84.

ELEMENTS OF MECHANICS.

Book III.

ON MOVING AND RESISTING FORCES.

—o—

CHAP. I.

ON THE MECHANICAL PROPERTIES OF BODIES.

413. Hitherto, we have supposed certain properties and certain forces to belong to matter and to bodies; and, according to these suppositions, we have investigated the conditions of equilibrium, and the laws of motion. Turning, now, to the terrestrial bodies, which surround us, we have to enquire, what are, in reality, their mechanical properties, and what is the measure and the nature of the forces, which urge them to motion: in which enquiry, observation and experience alone can be our guides. Following, therefore, these guides, we shall immediately begin by verifying the suppositions first made, of the impenetrability, and of the inertia of matter.

414. The impenetrability of matter is proved by an induction so extensive, that it may well be said to be universal. For we never see a body proceed to occupy the space first filled by another body, unless it has either driven the latter from that space, or else has insinuated itself into its pores; which pores are perceptible in all bodies, sometimes to the naked eye, and always by means of a microscope.

415. The property of inertia supposes two things; 1st, that a body does not pass from a state of rest to motion, without the intervention of some force; 2ndly, that when it has been put in motion, it invariably preserves its velocity, and its direction, unless some force intervene to change them. Now the first of these suppositions is manifestly true. For a body cannot begin to move without the operation of a cause of some kind; otherwise, there would be no reason why it should have one particular motion rather than another; and this cause is (9) called a *force*. The truth of the second supposition is evident from induction. For we never see either the velocity, or the direction, of a moving body altered, unless through the intervention of some one of those causes, which, from previous investigation, we know to have the power of impressing or of retarding motion; and we see the impressed motion maintain itself the longer, the more completely we succeed in removing these disturbing causes.

CHAP. II.

ON GRAVITY.

416. That gravity is a force common to all the elements of matter, is rendered sufficiently plain from observing, that every body, and every particle of matter, however small, when it is not supported, falls in a vertical direction. It remains to be proved, that this force is constant in any the same body, and equal in different bodies; and we have, further, to assign the measure of it. For which purpose we must shew, that the motion of heavy bodies, when they fall, is an equably accelerated motion; and that the value of the accelerating force is the same, whatever be the body which falls; and, in the last place, we must determine this value. But, in experiments of this kind, no small embarrassment arises from resistances. If we cause heavy bodies to fall *in vacuo*, the descents are too rapid to be compared with the times; if we let them drop from the tops of towers of considerable height, as was done by Riccioli and Grimaldi, we encounter the air's resistance, which increases with the increase of the velocity, and changes the nature of the motion; if we make them descend, as Galileo did, down planes of a gentle declivity, to the resistance of the air is added that of attrition. The experiments of the above-mentioned authors correspond, nevertheless, with the sufficient exactness, to the hypothesis of a constant gravity. The best method, however, of investigating the nature of the motion produced by the force of gravity, is that of observing the oscillations of a pendulum through very small arcs; in which kind of motion the resistance of the air does not increase beyond a certain limit, and we may easily contrive (296), so that it shall not sensibly alter the time of an oscillation.

417. *Experiment* I. The oscillations of a pendulum through very small arcs of a circle are constantly found (Gal. *Op.* tom. III. p. 56), to be all of them isochronal with each other; of which fact it is most easy to be assured, by counting the number of oscillations made in equal intervals of time.

418. *Coroll.* Gravity is, therefore, a constant accelerating force.

419. *Experiment* II. In different hollow spheres, of equal weight and diameter, are enclosed equal weights of different substances; these spheres are suspended, by strings of equal length, and are made to oscillate through very small arcs; and it is found (Gal. *Op.* tom. III, p. 49, 50.), that the time of an oscillation is the same in them all.

420. *Coroll.* Wherefore, the force of gravity is the same in different bodies.

For the time of an oscillation is expressed (284) by $t = \frac{\pi\sqrt{a}}{\sqrt{g}}$; whence $g = \frac{\pi^2 a}{t^2}$; and t being the same, whatever the body is, the force g must also be the same, whatever the body is.

This is confirmed by the well known experiment of the fall of heavy bodies *in vacuo;* for gold, and feathers, fall through equal altitudes in equal times.

421. *Experiment* III. The length of a pendulum, which vibrates seconds, is investigated and measured with very great precision; by which length is understood (354) the distance of the point of suspension from the centre of oscillation. This has been done, with incomparable diligence, at Paris, first by Mairan, (*Mem. de l'Acad. des Sc.* 1735), and afterwards by Borda, (*Instit. Nat.* tom. II, p. 79); and this length has been found 0.9938 metres, or 39.1267 English inches.

422. *Coroll.* 1. Wherefore, taking a metre for the unity of spaces, and a second for the unity of times, we shall have $g =$ 9.8088. For, in the formula $g = \frac{\pi^2 a}{t^2}$, if $t = 1$, and if $a = 0.9938$, then $g = 9.8088$.

423. *Coroll.* 2. A heavy body, falling freely, will describe, in the first 1″ of its descent, a space (210) $= \frac{1}{2}g = 4.9044$ metres, or 193.09 English inches.

424. Scholium. The experiments of the pendulum both furnish us with the most simple and the most secure method of confirming the hypothesis of an equal and constant gravity, and are, at the same time, the best adapted to indicate any exception, to which that hypothesis may be liable. It is very easy to discover any the slightest alteration in the motion of a pendulum; because the amount of the increase or decrease of its rate of going is accumulated, as it were, in a great number of oscillations, and soon shews itself. Thus, it is found (Newt. *Princ.* Lib. III. Prop. 20.), that the force of gravity is different, in different latitudes, and goes on increasing from the equator to the poles. And, in very great elevations above the level of the sea, a certain diminution of gravity, (Bouguer, *Fig. de la Terre*, p. 357.) has been similarly detected. But in practical Mechanics, where the comparison is only between bodies that are not far distant from each other, we may neglect all these anomalies.

CHAP. III.

ON ELASTICITY.

425. For the elasticity of bent laminas, we have assumed this hypothesis; that, in every point of the curved lamina, the momentum, with which the elasticity tends to make an element of the lamina unbend itself, so as to be in the same direction with the contiguous element, is reciprocally proportional to the radius of curvature.

Let AL, LB, (Fig. 38.) be two successive equal elements of the bent lamina ALB; of which the second LB, if the force that bends the lamina were removed, would take the position Lb, in the direction of the first AL. Join B, b, and draw LP perpendicular to Bb, which will bisect Bb in P, the triangle BLb being isosceles.

The force, with which the point B endeavours to transport itself to b, may be considered as proportional to Bb, that is, to the distance through which the point B was removed from its natural position; and, because this force acts in the direction Bb, its momentum, with respect to the point L, will be expressed by $E.Bb.LP$, E being a constant quantity. It remains to be proved, that this expression is reciprocally proportional to the radius of the circle, which passes through the three points A, L, B.

Putting $AL = LB = Lb = 1$, and the angle $BLb = 2\omega$, there results $Bb = 2 \sin. \omega$, and $LP = \cos. \omega$; whence $E.Bb.LP = 2E.\sin. \omega.\cos. \omega = 2E.\sin. \omega$ nearly, because the angle ω being indefinitely small, $\cos. \omega$ may be considered as equal to unity.

Again, let K be the centre of the circle, which passes through the three points A, L, B, let the radius $KL = R$, and from the point K let fall on LB the perpendicular KQ, which will bisect it in Q. Then the angle $LKQ = \omega$, and

$$1 : \sin.\ \omega :: KL : LQ :: R : \tfrac{1}{2};$$

$$\text{wherefore } \sin.\ \omega = \frac{1}{2R}.$$

This value having been substituted, the force of elasticity becomes $\frac{E}{R}$, according to the hypothesis (192).

426. Hence it appears, that the proposition of Art. 192, is founded on the supposition that the force, tending to restore the elastic lamina, is proportional to the quantity by which it has been bent from its primitive situation. But although it is true, and may be easily proved by experiment, that the further the elastic lamina is drawn from its natural position, the greater is the force with which it restores itself; it is not, however, perfectly well ascertained, that the force of restitution is exactly proportional to that of traction.

The same must be said of the other hypothesis (398), which we have followed, in calculating the effect of percussion amongst elastic bodies. We have assumed, that the force with which they restore themselves, is always a determinate part of the force which causes the compression. And it is true, that the more an elastic body is compressed, the greater is the force, with which it endeavours to restore itself; but it is not equally certain, that the force of restitution is always proportional to the force of compression.

427. For bodies which suffer compression from impact, an easy method of measuring the degree of elasticity, and, at the same time, of ascertaining whether it is proportional to the compressing force, would be to strike them directly against an immoveable plane, with different velocities, and to measure the velocity of the rebound. If the force of elasticity be proportional to the compressing force, the velocity of reflection will be (405) in a constant proportion to the velocity of incidence; and this proportion will determine the ratio of the force of elasticity to that of compression.

428. The motion, with which the particles of an elastic body restore themselves to their first situation, is an accelerated motion;

not, indeed, equably accelerated, but by degrees always less and less; because the impulses of elasticity go on (426) diminishing in proportion as the body is less and less compressed.

429. The velocity, acquired at the end of this motion, is preserved, when the body is free; but if it encounter an obstacle, every particle will preserve that portion of it, which is not destroyed by cohesion and other impediments. Hence, an elastic body, stopped in a point, and compressed or bent, after its restitution cannot rest there; and, if the cohesion of its particles suffer it, it will pass from its primitive state to the opposite state; between which two states it will go on oscillating, so long as the resistances do not stop it. Thus a lamina of steel, fixt horizontally in a wall, and bent downwards by a force, springs, the moment that it is released, as much upwards, and vibrates, for a long time, like a pendulum.

430. An elastic body, compressed, tends to dilate itself with an equal force, in every direction, and does, in reality, so dilate itself, when it is not prevented by the cohesion which binds together its particles. Hence, wherever there is no cohesion, as in elastic fluids, the effort of dilatation is exerted equally on every side.

CHAP. IV.

ON THE ELASTICITY OF THE AIR.

431. The elasticity of aeriform fluids is a force of singular energy; the effects, which it produces, are wonderful; and human industry, applying it to mechanical purposes, derives from it very great advantage. It is, therefore, of great importance to know, and to measure with precision, this force, in those fluids of which we can most conveniently make use. Such are the air, and aqueous vapours.

432. The elasticity of fluids exerts itself with equal force (430) in all directions: wherefore, the pressure of an elastic fluid upon a given base, is always proportional to the surface pressed. If we imagine a prismatic column, of a known substance, of mercury, for example, which insists perpendicularly on the given base, and, pressing it with its whole weight, is in equilibrium with the force of the elastic fluid, the altitude of this column will measure the force, which the fluid exerts against the base, by its elasticity. Mercury is chosen, in order to compare the elasticity of different fluids with that of the air in its natural state; which, as is well known, is measured by the mean altitude of the barometer; that is, by the altitude of a column of mercury of 0.76 metres.

433. *Experiment* I. In order to discover according to what law the air's elasticity increases, whilst its density increases, Mariotte (*Mouvem. des Eaux*, Part II, Disc. 2), made use of a long syphon, having its two cylindrical vertical branches connected by a short horizontal tube. The branches were of unequal length; the shorter hermetically closed at its top, the longer open. Through this he poured a little mercury, just sufficient to fill the horizontal tube, and to confine the air in the shorter branch. He then went on pouring in more and more mercury, and noting, at every time, the level at which it stood in both the branches. Thus he was enabled easily to discern the pro-

portion in which the density, and the elasticity of the air included in the shorter branch, went on increasing; because the density necessarily increases in the inverse ratio of the spaces into which the air was successively crowded; and the elasticity was measured by the difference of the levels of the mercury in the two branches; there being added to it 0.76 metres, for the weight of the atmosphere pressing on the longer branch.

434. *Coroll.* The result of many trials, made in the above-mentioned manner, was this,—that, under the same temperature, the elasticity of the air is proportional to its density.

435. SCHOLIUM. This proportionality, which is found to obtain in the mean compressions of atmospheric air, cannot, with equal certainty, be ascribed also to the greatest and least compressions. For in these no trial has been made; besides that the compressing weight may vary indefinitely, but the density cannot vary indefinitely.

436. *Experiment* II. In order to discover how the air's elasticity increases, whilst its temperature increases, the experiments of Volta, (*Ann. di Chim. di Brugnatelli,* tom. IV.) upon the dilatation of the air by heat, are extremely well contrived. Having taken a hollow sphere of glass, which ended in a straight cylindrical tube, minutely graduated, and having carefully measured its capacity, he filled it partly with oil, the rest remaining full of air. Then, closing the orifice of the tube with his finger, and inverting it, he immersed it in a large vessel full of oil. Thus the included air rose to the top, occupying the sphere, and the upper part of the tube; and, in addition to the pressure of the atmosphere, it sustained that of the oil, corresponding to the altitude of the level of the vessel above the lowest boundary of the air in the tube. He caused the included air to pass through every degree of temperature, from that of ice to that of boiling water; which he effected, by heating, gradually, the oil of the surrounding vessel. He observed, from time to time, how much the air, included in the tube, expanded itself; and, as the pressure was always the same, he took care to go on elevating the tube, at every observation, so that the lowest boundary of the air always remained at the same depth below

the level of the vessel. He thus found that, under the same pressure, for every degree of increased temperature, as indicated by Reaumur's thermometer, the volume of the air uniformly increased by $\frac{1}{215}$ of the volume, which it originally had, at the temperature of zero: which amounts to $\frac{1}{268}$, for every degree of the centigrade thermometer, with which, for the future, we shall measure temperatures.

437. Scholium 1. Some time after the experiments of Volta, above described, Gay Lussac (Biot, *Traité de Phys.* Lib. I. Ch. IX.) aud Dalton, with different kinds of apparatus, and by different modes of operating, have confirmed the same results; that is, that the expansion of the air, by heat, is uniform, and that it amounts to $\frac{1}{268}$, for every degree of increased temperature. Further, they have found that all the permanent gaseous fluids are expanded, by heat, in the same manner as air is. But we cannot deny to Volta the praise of having been the first to establish, with certainty, this point; which was much disputed amongst the cultivators of Natural Philosophy, from whose experiments arose a very great variety and difference of opinions. Almost all of them found the expansion of the air to be not uniform; and some of them made it considerably greater, others less. Volta shewed that these discrepancies arose from moisture, which was not excluded, as it ought to have been. Operating upon air, that was perfectly dry, all the anomalies disappeared, and the true law and measure of the expansion of air, by heat, became manifest.

438. *Coroll.* 1. Hence, we infer, that, the density remaining the same, the elasticity of the air, taken at the temperature of ice, increases by $\frac{1}{268}$ of its value, for every degree of increased temperature. For, in the experiment above related, when t degrees were added to the temperature, the primitive elasticity ought to have decreased (434) in the ratio of $1 + \frac{t}{268} : 1$; in

consequence of the density having been diminished in the same ratio. But the elasticity remained the same. Wherefore, it increased as much, by the augmentation of temperature, as it ought to have decreased by the diminution of its density, that is, by $\frac{t}{268}$ of its primitive value.

439. *Coroll.* 2. Let P be the air's elasticity when the density $=1$, at the temperature of 0^0; let p be its elasticity, when the density $=q$, and at the temperature of t: then,

$$p = Pq\left(1 + \frac{t}{268}\right).$$

440. Scholium 2. This proposition, must likewise be restricted by the limits of temperature, to which experiments have been extended; that is, by the temperatures of ice, and of boiling water. Beyond these limits, the variations of elasticity, caused by heat, may perhaps follow other laws.

441. Scholium 3. The air spoken of must also be understood to be perfectly dry; air, loaded with moisture and with vapours, does not expand equably: equal increments of temperature are not accompanied by equal expansions, but always by greater. Whence we may conclude, that, in this case, the elasticity does not increase in the same proportion as the temperature, but in a greater proportion: of which we shall very soon have a direct confirmation.

CHAP. V.

ON THE ELASTICITY OF AQUEOUS VAPOURS.

442. It is most certain, that the elasticity of vapours, and that of air, depend upon the same elements; upon density and temperature. Hence, it would appear, that, both in the case of air, and in that of vapours, we ought to divide these two elements, and to investigate, separately, the effects of each; by making the density vary, whilst the temperature remains constant; and *vice versâ*. But we are freed from this trouble, by a singular property of vapours, not long since discovered, and fully ascertained by the most accurate experimenters: namely, that their density necessarily depends upon their temperature; so that to every degree of temperature belongs a certain and determinate degree of density, which remains constant, whilst the space through which the vapour is diffused, is diminished or increased. And this happens, because, whilst the space occupied is diminished, a part of the vapour is condensed, and is turned into water; and whilst the space occupied is dilated, the subjacent water disengages a fresh supply of vapour; whence upon the whole, the density suffers no change.

443. This constant density, however, of vapour only obtains, where there is a quantity of water, sufficient to furnish so much vapour as will fill the whole capacity of the space, through which the vapour is diffused, and will maintain it at the assigned degree of density. Otherwise, the density will necessarily decrease, and will go on decreasing more and more, the more the capacity of the recipient is enlarged: and, on the contrary, if this capacity be diminished, the density will go on increasing, until it reach the assigned degree; at which it will remain constant, however the space be afterwards diminished.

When vapour is employed as a mechanical agent, there is always a reservoir of water, capable of evolving new vapour sufficient to keep the recipient space full. Wherefore, we may safely suppose vapour to have that degree of density, which belongs to its temperature, and we may investigate how much its elastic force increases, whilst its temperature increases.

444. *Experiment.* This investigation has been made, with singular accuracy (*Manchester Mem.* 1805. Biot, *Traité de Phys.* Lib. I. Ch. XIII.) by Dalton. His apparatus consisted of a simple barometer-tube, of which the inner surface was wetted, before the introduction of the mercury. The mercury, well freed from air, having been afterwards poured in, after the usual manner, the tube, having been inverted, and its mouth having been immersed in a vessel full of mercury, the moisture, introduced into the tube was collected in the vacuum at the top, and a thin layer of water stood upon the surface of the mercury. The temperature was then gradually increased, by pouring water, more and more heated, into a gun-barrel which surrounded the whole of the upper part of the tube. As the temperature increased, the column of mercury sunk more and more. The height of this column having been subtracted from the height, which represents the pressure of the atmosphere, that is, from the height of the mercury in a common barometer, there resulted the measure of the elasticity of the vapour.

445. Scholium 1. The column of mercury remaining at the height, which corresponds to the degree of the temperature, experiments were made, of lowering the tube, by plunging its orifice to a greater depth in the subjacent vessel of mercury; and also of raising it; and it was found, in both cases, that the height of the column of mercury, above the level of that in the vessel, always remained the same. This is a plain proof, that the elasticity of the vapour also remains the same, although the capacity of the space occupied by the vapour is diminished in the first case, and increased in the second. And this confirms the position that, at the same temperature, the density of the vapour remains constant, as (442) was said above.

446. *Coroll.* 1. The correspondence between the temperature of the vapour and its elastic force, is as follows:

Temperature.	Elasticity Metres.
0^0	0.005
10	0.009
20	0.017
30	0.031
40	0.053
50	0.088
60	0.145
70	0.228
80	0.352
90	0.525
100	0.760
110	* 1.069
120	* 1.462
130	* 1.941

It is to be remarked that, with Dalton's apparatus, the temperature could not be carried higher than that of boiling water: hence, the elasticities, corresponding to temperatures greater than 100^0, were not found from observation, but from calculations made upon the supposition that they go on according to the same law, as that which they follow in lower temperatures; which law we shall proceed to investigate.

447. *Coroll.* 2. The temperatures increasing in arithmetic progression, the elasticities increase nearly in geometric progression. And it will be found that, in fact, their logarithms increase with very nearly equal differences.

448. *Coroll.* 3. But these differences of the logarithms, although they differ but little from each other, do not remain precisely equal, but go on slowly decreasing, as the temperature is raised. Whence we may infer, that the elastic forces increase in a somewhat less proportion, than that which is indicated by the geometric progression.

449. *Coroll.* 4. Laplace (*Mec. Cel.* tom. IV. p. 272.) has given a formula which represents, with great exactness, the

results of the experiments of Dalton, and thus exhibits the true law of the increments of the elasticity of vapour. If t denote the temperature, and p the corresponding elasticity, then,

$$p = 0.76 \times 10^{k(t-100) - m(t-100)^2},$$

where $k = 0.0154547$, $m = 0.0000625826$.

450. Scholium 2. With this formula, the elasticities belonging to each degree of temperature are readily computed; for it will be sufficient to add the number $k(t-100)-m(t-100)^2$ to the logarithm of 0.76, in order to obtain the logarithm of p.

451. Scholium 3. Bettancour has, also, with great diligence, succeeded in investigating the elasticity of vapours. His apparatus consists of a large vat, or boiler, a determinate portion of which he fills with water; he then closes it hermetically, and exhausts the air; afterwards, applying fire below it, he gradually raises the temperature of the water, and of the vapour which is disengaged from it and diffused through the vacuum of the boiler. A long barometer, in the form of a syphon, communicates with the interior of the boiler. In proportion as the vapour, more and more heated, acquired force, the level of the mercury fell in the nearer branch of the syphon, and rose in the other branch. The difference of the levels measures the elasticity of the vapour, at the same time that a thermometer, with its bulb plunged in the boiler, and rising out of it with a long neck, indicates the corresponding temperature.

The results of Bettancour's experiments, although they do not exactly coincide with those of the experiments of Dalton, are, however, sufficiently conformable to them. And thus the law of nature, which establishes a correspondence between the temperature and the elasticity of aqueous vapour, is fully ascertained, by two series of experiments, conducted according to different processes.

452. Scholium 4. Since the elasticity increases much faster than the temperature, we see plainly that, in very high temperatures, the force of vapour may become surprisingly great; and the prodigious effects, which are related of it, become credible.

CHAP. VI.

ON THE FORCE OF GUNPOWDER.

453. The violent elasticity of gunpowder, set fire to in a confined space, is that of the vapours of water, of the vapours of nitric acid, and perhaps of other aeriform fluids, which, in the combustion of the powder, are suddenly disengaged in great abundance. Their force depends solely upon the density; because the temperature is constant: and the density varies according to the proportion of the quantity of powder to the space in which it is shut up. Hence originates the problem, to find in what proportion the elasticity of the vapour of powder increases, when the density is increased in a given ratio. We shall relate the latest experiments, tried with this view, by Count Rumford. (*Phil. Trans.* 1797.)

454. *Experiment*. In this experiment a small iron cannon was made use of, having strong and solid sides, and an internal capacity of 1472 cubic millimetres. First, a very small charge of powder, weighing 0.061 grams, was put in; which charge filled 0.039 parts of the internal capacity. The area of the mouth of the gun was 31.66 square millimetres; over it was placed a plane, which exactly closed it, and this plane was loaded with a heavy weight. In this capacity, entirely closed up, the powder is set fire to from without, by the heat of a red-hot iron, applied to a protuberance made in the inner surface, for that purpose. In this manner, Count Rumford instituted a series of trials, adding, by little and little, to the weight of the plane, which closed the mouth of the gun, until he found a weight which was hardly moved at all by the discharge. He then repeated the experiment, introducing into the tube a double quantity of powder, a triple quantity, and so on. Thus, the density of the vapour increasing in arithmetic progression, he had, at each trial, the corresponding elasticity, measured by the greatest weight, which the force of the vapour could move.

455. *Coroll.* 1. The results of these experiments were as they are given below; where the first column shews the proportion in which the densities increased; the second indicates the corresponding elasticities, expressed in the manner explained in Art. 432. For which purpose, we have reduced each of the observed weights to the weight of a column of mercury standing upon the area of the mouth of the tube, and we take the height of this column for the measure of the elastic force.

Density.	Elasticity.
1	59[m]
2	100
3	219
4	290
5	426
6	
7	617
8	885
9	1179
10	1432
11	1686
12	1956
13	2499
14	3046
15	3589
16	5388
17	
18	8342

456. *Coroll.* 2. The elasticity does not, therefore increase in the proportion of the density, as Robins thought, but in a proportion much greater.

The law of this increase is not easy to be found; in consequence of the irregular progress of the series of numbers, which represent the elasticities: nor do count Rumford's attempts seem well adapted to such an object: besides that the results of such experiments as these do not admit of great precision.

457. *Coroll.* 3. The opinion of Robins, that the greatest force of gunpowder, corresponding to the greatest degree of

density, is only a thousand times greater than the pressure of the atmosphere, is also false. We here find it to be considerably greater, and we are still very far from the greatest possible density. We have no experiments, which can indicate the highest degree of this force.

458. Scholium 1. If, in the discharges of pieces of artillery, the powder acted with all the force of which it is capable, there would be no piece so strong, as not to burst at the first discharge. If this does not actually take place, it is, first, because the ball does not exactly fill the bore of the piece; and secondly, because the powder, though it is rapidly ignited, is not instantaneously ignited; whence it happens, that, as soon as the portion of powder, first ignited, has removed the obstacle, the rest, taking fire in a less narrow space, exerts a less degree of force. Hence it may be understood, how difficult it is to determine, theoretically, the force with which the ball is shot.

459. Scholium 2. Wherefore, the best means of strengthening, where it is necessary, the force of the charge, are directed towards causing that which is used for wadding exactly to shut in the charge, and to yield slowly, at the first burst of the powder; giving it time to be ignited throughout, before the space occupied by it is much enlarged. Upon this depends the efficacy of the practice, lately introduced, of blasting rocks by gunpowder; which consists (*Encyclop. Brit. Suppl.* Vol. II. Part 2.) in closing the channel to the mine with fine sand, without ramming it at all.

CHAP. VII.

ON THE FORCE OF ANIMAL AGENTS.

460. So great and so variable are the elements, which concur in modifying the force of animals, that it becomes most exceedingly difficult to reduce them to any certain measure. We shall endeavour to draw from observation and experiment, the surest indications with which they can supply us upon this subject. But, in order best to regulate our researches, it will be of advantage, in the first place, to distinguish, and to class, the principal causes, which are capable of producing variations in the value of such forces.

461. First, then, the force varies, not only in different species of animals, but, also, in different individuals. And this variation depends, first, on the particular constitution of the individual, and upon the complication of causes, which may influence it; secondly, upon the particular dexterity acquired by habit. It is plain, that such a variation cannot be subjected to any law, and that there is no expedient to which we can have recourse, but that of seeking mean results; thus deducing, from multiplied trials, the force of individuals of mean power.

462. Secondly, the force varies according to the nature of the labour. Different muscles are brought into action in different gestures of an animal which labours; the weight itself of the animal machine is an aid in some kinds of labour, and a disadvantage in others: whence it is not surprising that the force exerted is different, in different kinds of work. Thus the force exerted by a man is different, in carrying a weight, in drawing or pushing it horizontally, and in drawing or pushing it vertically. To these three modes of action which are employed sometimes separately, sometimes conjointly, it appears that all those which ordinarily occur in mechanical labours, may be reduced.

463. Thirdly, the force varies according to the duration of the labour. The force, for example, which man can exert in an effort of a few instants, is different from that which he can maintain equably in a course of action continued, or interrupted only by short intervals, for a whole day of labour, without inducing excessive fatigue. The former of these, then we shall call *Absolute Force*, the latter *Permanent Force*. It is of use to become acquainted with them both, as it is often advantageous to avail ourselves sometimes of the one, sometimes of the other.

464. Lastly, the force varies according to the different degrees of velocity, with which the animal, in the act of labouring, moves either its whole body, or that part of it which operates. The force of the animal is the greatest, when it stands still; and grows weaker as it moves forward, in proportion to its speed; the animal acquiring, at last, such a degree of velocity as renders it incapable of exerting any force.

465. Several writers (De la Hire, *Mem. de l'Acad. de Paris*, 1699. Lambert, *Mem. de l'Acad. de Berlin*, 1776) have very ingeniously attempted to deduce directly, from a consideration of the animal machine, and of its disposition in different gestures, the measure and the variation of its force. But, too many data are wanting to the attainment of this end. Hence, they have been under the necessity of assuming various suppositions, almost arbitrarily: and thus, even when the last result is comformable to experiments, no certain conclusion can be drawn from it, in confirmation of the assumed principles, and of the justness of the theory. It appears better, therefore, to have recourse immediately to observations and experiments, in order to deduce from them, if it be possible, the measure of the force of animals, and those modifications, which the circumstances, above pointed out, produce in it.

CHAP. VIII.

ON THE ABSOLUTE FORCE OF MAN.

466. The dynamometer of Regnier is employed with advantage, in discovering the absolute force of man, under different modes of action. This instrument consists (*Journal Polytech.* Ch. V.) of an elliptical spring, which is bent, either by pressing against each other the two vertexes of the axis minor, or by drawing, in contrary directions, the two vertexes of the axis major. In both cases, the sides of the spring are made to approach each other; and thus they move an index, the point of which, describing a semi-circle, marks the degree of approximation.

The semi-circle has two divisions; the one, for the case, in which the vertexes of the axis minor are pressed toward each other, the other for the case, in which those of the axis major are stretched in contrary directions.

The first division is made thus: the dynamometer is placed with its axis minor vertical; and the lower extremity of this axis having been firmly fixt, its upper vertex is successively loaded with weights increasing by a chilogram at a time; and the points, at which the index successively stops, mark the degrees corresponding to those weights. The second division is completed in the same manner; by placing the dynamometer with its axis major vertical, and with its upper vertex firmly fixt, and then by loading the lower vertex with weights increasing by equal differences.

By the help of this machine, the mean force of man is ascertained, first, in pressing a body with one hand, or with both hands; secondly, in drawing and raising a weight vertically; thirdly, in drawing a weight horizontally. These three experiments give the following results.

467. *Experiment* I. To try the force of the hand, the dynamometer is grasped towards the axis minor, and pressed as strongly as is possible: the index marks, on the first graduation, the weight equivalent to the force exerted in the pressure. The mean measure of this force was found by many trials, to amount to 50 chilograms.

468. Scholium. Some curious particularities were observed in the performance of these experiments. The most convenient and advantageous attitude for pressing the instrument, is to put forward the arms, giving them an inclination to the vertical of half a right angle. The right hand commonly presses with greater force than the left: and the force, which both hands exert together, is equal to the sum of the forces which they exert separately.

469. *Experiment* II. To try the force of man in raising a weight vertically, the dynamometer is placed with its axis major vertical; the lower vertex is attached to the ground by a hook; to the upper vertex, by means of another hook, is attached an horizontal handle, which the man takes hold of with both hands, and draws straight upwards, with all his strength. Then the index marks, on the second graduation, the weight equivalent to the force exerted. The mean measure of this force turns out to be 130 chilograms.

470. Scholium 1. Here the most advantageous position is for the man to keep his body straight and vertical, except that he inclines the shoulders a little forward. And this position is of great consequence; for thus he can sustain a much greater weight than in any other attitude.

471. Scholium 2. Although it is not ascertained by the dynamometer, either what force a man exerts, when, instead of drawing vertically upwards, he sustains, or raises, a weight, by drawing, or by pushing, vertically downwards, or what force he exerts, when he carries a weight upon his shoulders, it seems probable, however, that the mean measure of these forces must be somewhat greater, but not much greater, than that indicated above. The greatest weight, which a man of mean strength, standing still, can support for some time, is commonly reckoned to be 150 chilograms.

472. *Experiment* III. To try the force of man in drawing horizontally, the dynamometer is managed as in the preceding experiment, except that it is placed with its axis major horizontal. The result of many trials, made in this way, is that the mean force of a man, when he draws horizontally, in the position usually taken by those who draw hand-carts, or boats, is 50 chilograms.

473. Scholium. In different individuals the absolute force varies much, as well in the action of pressing, as in that of drawing vertically. It is not so, in drawing horizontally; in which exertion, the difference of force is circumscribed within narrow limits, compared with those of the other two. The force of the most robust men, in the horizontal draught, is scarcely greater than 60 chilograms, thus exceeding, by only 10 chilograms, the mean force. The reason of this remarkable difference may perhaps be, that, in the horizontal draught, a man assists himself more by his weight, than by the force of his muscles; whilst the other exertions depeud entirely upon muscular force. Now, however different the weight may be, in different men, this variation is incomparably less than that which natural constitution, and exercise, are accustomed to produce in the muscular force of different individuals.

CHAP. IX.

ON THE PERMANENT FORCE OF MAN.

474. To acquire exact and sound notions upon the measure of the permanent force of man, there is no better method than to make a series of observations upon the long continued labours of those who are paid by the day for their work, and to note the force which they exert, the speed with which they advance, the duration and the intermissions of their toil. The trials, that are made, by causing a few workmen to labour for a short time, and for the purpose of being observed, are little, or not at all, to be relied upon. A man may easily, for a short time, force his labour; and he never fails to do so, when he is aware that he is observed, and that it is wished to make a trial of his strength. We ought, therefore, to abandon such attempts, and to seek for information, rather from what we observe, than from set trials. Hardly, however, has any one applied himself to observations of this kind, although whosoever has the opportunity, may easily make them. Coulomb (*Instit. Nat.* tom. II.) has collected some of these observations, with great care and sagacity; and we shall here report such of them as refer to the most common mechanical labours.

475. But first, it is proper to assume a certain and constant measure, to which we may refer this permanent force. The absolute force, being exerted by man whilst he stands still, has for its natural measure the weight which he can support or move; but the permanent force, being exerted whilst the man walks forward, or moves some part of his body, it becomes necessary in this case, to take account, not only of the weight moved, but also of the velocity with which it is moved. We shall estimate, therefore, the *effect* of the permanent force, by the product of the weight and its velocity.

Afterwards, when we seek the effect of the permanent force, for a whole day of labour, it is clear that, in comparing the per-

manent force, exerted in different kinds of work, whilst the time of working remains the same, the velocity is proportional to the space passed over. Wherefore, in making this comparison, we may also take, for a measure of the effect, the product of the weight by the space, through which it has been moved, in the entire labour of a day.

476. *Experiment* I. Observations having been made upon the toil of different porters, employed to carry goods to a distance of 2000 metres, each of them was found to carry, in the course of the day, 348 chilograms, at six journies, loaded with 58 chilograms at a time.

Here the effect is expressed by $348 \times 2000 = 696000$.

477. Scholium. The neat effect becomes greater, when the man, loaded with a less weight, goes straight on, without alternately loading and unloading, and without the interruption of returning empty. Coulomb, having interrogated several of those pedlars, who travel loaded with their packs, collected from their answers, that with a weight of 44 chilograms, they can travel 19000 metres in a day. The neat effect is, therefore, 836000. Although there may be some exaggeration, as Coulomb suspects, in this assertion, this kind of action does, nevertheless, appear to have a sensible advantage above the former.

478. *Experiment* II. In like manner, other porters carried, in a day, 4404 chilograms of wood each, mounting a convenient staircase 12 metres high. They made, in the course of the day, 66 journies, and carried, at each time, 66.7 chilograms, or, including the weight of the cords and clasps, about 68 chilograms.

Here, justly to measure the effect, we ought to know precisely the inclination of the staircase; because we have to multiply the weight, not by the vertical height, to which it was raised, but by the length of the space described. Coulomb elsewhere supposes the height of a convenient staircase to be one-third of its horizontal base. If this was so, the length of the journey was 37.95 metres; and the neat effect was $68 \times 66 \times 37.95 = 170320$ nearly. Upon any supposition it is evident, that the neat effect is much less in this kind of work, than it is in the carrying of goods on the plane ground; a very natural result,

when the weight of the man himself is taken into account, which must be raised along with the weight. The portion of the day, actually employed in labour, was six hours and a half, with many interruptions: for each ascent of a man was completed in about 1.1 minutes. Thus the duration of the fatiguing action did not exceed 1 hour 12 minutes in the day; and the velocity of ascent was 0.575 metres.

479. *Experiment* III. The ordinary process of driving piles, according to the observation of Coulomb, is as follows:

Such a number of men commonly work at raising the ram, as that each lifts 19 chilograms of it. At every pull the ram is raised 1.1 metres. They make twenty pulls in one minute, and after three minutes of exertion, they rest for as long a time, and then begin again. Thus, their daily labour lasting six hours, the duration of fatiguing exertion does not exceed three hours. This is, notwithstanding, a sufficiently painful toil; nor could it be carried on during many days, without the loss of health, and exhaustion of strength.

Here, the neat effect is $19 \times 1.1 \times 3600 = 75240$.

The following observation is by Lamandè, architect of the bridge of Jena, (Hachette, *Traitè des Mach.* p. 10).

Thirty eight labourers, working ten hours a day, make in each hour 12 efforts, each of 30 pulls; hence, exactly as in the observation of Coulomb, they make 3600 pulls in a day. The ram weighs 587 chilograms, and is raised 1.45 metres.

According to this observation, the neat effect will be $\frac{587}{38} \times 1.45 \times 3600 = 80635$ nearly; a result not very different from that of Coulomb.

480. *Experiment* IV. Workmen having been observed, for a long time, employed in turning a lever, Coulomb relates, that under a continued labour, they exert a force equivalent to 7 chilograms, and make 20 turns in a minute, the circumference of the turn being 2.3 metres. They continue at work for eight hours; but, owing to the rests, of which they have need, from time to time, the duration of the daily fatigue does not exceed six hours.

Wherefore, the neat effect will be $7 \times 2.3 \times 7200 = 115920$.

481. Scholium. By some other trials, in this kind of work, (Desagulier's *Cours de Phys.* Lect. IV.) a much greater effect has been found. In these, there were counted at least eight hours of effective labour, at 30 turns in a minute, and with a continued force of 12.2 chilograms. The resulting effect would, therefore, be 404064. But we have already (474) observed how deceitful are such trials; here we have a confirmation of it.

482. *Experiment* V. In the conveyance of earth by wheelbarrows, it happens, that the weight of the barrow is ordinarily about 30 chilograms, and that it is loaded with 70 chilograms; the labourer, however, does not carry more than from 18 to 20 chilograms, the remainder being supported by the ground. At the same time, he pushes with a force, which may be estimated at from two to three chilograms; this degree of force being sufficient to overcome the attrition of the barrow on a dry plane, although the irregularity of the soil, and the dexterity of the workman, render this force sometimes greater, sometimes less. He afterwards returns, by the same track, with the barrow empty; and he then does not carry more than from five to six chilograms, and a very little force is sufficient for him to urge the barrow forwards. In this manner, he can easily make, in one day, 500 journies of 29.226 metres each, and as many returns.

483. Scholium I. This is a mixt action; for the labourer supports a weight, and, at the same time, pushes horizontally. To measure the effect, Coulomb multiplies the 70 chilograms of earth by the distance through which they are carried, which is 14613 metres; and the product is 1022910.

But this is not, in my opinion, an adequate measure of the effect. He who draws a weight along an horizontal plane, does not feel, nor overcome, the force of the weight itself; but it is a force equivalent to that of the friction, which he has to surmount. In like manner, to measure the effect of the force of the labourer in the 14613 metres, through which, between the weight, which he sustains, and the push which he makes, he exerts a force of about 22 chilograms; we ought to multiply 14613 by 22; and the effect will be 321486. Also, since, in returning, he

describes the same space, exerting a force of about 5.5 chilograms, the effect of this part of the exertion will be $14613 \times 5.5 = 80371.5$. Combining the two effects, the total effect of the daily exertion will be 401857.5.

484. Scholium II. It was the opinion of the celebrated Daniel Bernouilli, (*Prix. de l'Acad.* tom. VII.) that the measure of the effect of the permanent force of man, in his daily labour, is nearly a constant quantity; and that it does not much vary, either betwixt one individual and another, or betwixt one kind of labour and another. He expresses its mean measure by the number 274771. Now, the very great differences which we have found to obtain, between the effects of different exertions, shew the falsehood of this conjecture; for the effect of daily exertion is, manifestly, sometimes greater, sometimes very much less, than the measure assigned for it by Bernouilli.

CHAP. X.

ON THE RELATION BETWEEN THE FORCE AND THE VELOCITY OF THE AGENT.

485. Let g be a weight equivalent to the force, which a man can exert, standing still; and let h be the velocity with which, if he proceeds, he is no longer capable of exerting any force: also, let F be a weight equivalent to the force, which he exerts, when he proceeds, equably, with a velocity v.

Then, (464) F will be a function of v, such that, 1st, it decreases whilst v increases; 2ndly, when $v = 0$, then $F = g$; 3rdly, when $v = h$, $F = 0$.

486. Upon the nature of this function, instead of experiments or of observations, which alone could make it known, we have but the different suppositions of different writers, proposed without any other recommendation than that of their simplicity. They are the three following:

1. $F = g.\left(1 - \frac{v}{h}\right)$. (Bouguer, *Man. des. Vais.*

2. $F = g\left(1 - \frac{v^2}{h^2}\right)$. (Euler, *Nov. Comm. Pet.* tom. III).

3. $F = g\left(1 - \frac{v}{h}\right)^2$. (Ib. tom. VIII).

487. *Coroll.* 1. The effect of the permanent force being measured (475) by the product Fv, the expression for the effect will be one of the three following, accordingly as one or other of the suppositions is adopted.

1. $Fh.\left(1-\frac{F}{g}\right)$, or $gv\left(1-\frac{v}{h}\right)$.

2. $Fh.\sqrt{\left(1-\frac{F}{g}\right)}$, or $gv\left(1-\frac{v^2}{h^2}\right)$.

3. $Fh.\left(1-\sqrt{\frac{F}{g}}\right)$, or $gv\left(1-\frac{v}{h}\right)^2$.

488. *Coroll.* 2. To know the weight, with which a man should be loaded, or the velocity, with which he ought to move, in order to produce the greatest effect, we must make $d\,.\,Fv=0$. Whence, we shall have

1. $F=\frac{1}{2}g$; and $v=\frac{1}{2}h$.

2. $F=\frac{2}{3}g$; and $v=\frac{1}{\sqrt{3}}.h=0.58\,h$.

3. $F=\frac{4}{9}g$; and $v=\frac{1}{3}h$.

489. *Coroll.* 3. And the value of the greatest effect will be,

according to the 1st hypothesis .... $=\frac{1}{4}gh$;

..............2nd .... $=\frac{2}{3\sqrt{3}}gh=0.39gh$;

............3rd .... $=\frac{4}{27}gh$.

490. Scholium. But which of the three suppositions ought we to prefer? Are we certain that any of them approximates to the true law of nature? and, even if one of them may be considered as true, how shall we determine, for each particular kind of permanent labour, the values of the coefficients g, h? Euler assumes $g=34$ chilograms, $h=1.95$ metres; but this is altogether a precarious determination.

Reduced, up to the present time, to notices that are too vague and insecure, we should at least follow a track, which, sooner or later, may lead us to the acquisition of such as are more certain. It will conduce to this end, if, in each of those kinds of labour,

where it is of most importance to know the force of man, a series of observations be instituted, after the manner, and with the precautions, indicated in the preceding chapter; comparing together those cases, in which the labourers raise different weights, and noting, for each weight, the degree of velocity which they naturally take, and the effect thence produced. This series of observations would lead to the discovery of the form of the function, which expresses the force in terms of the velocity; and a few of the experiments would serve to determine the constant coefficients. Such a discovery would be of the greatest usefulness to the science of Mechanics, upon which it depends how to employ, to the greatest possible advantage, the force of animal agents.

CHAP. XI.

ON THE FORCE OF BEASTS.

491. If we know little, with precision, relative to the force of man, we know still less concerning the force of beasts of draught or of burden; which, considering the facility of collecting useful information, upon this point, and the great importance of the subject, for the direction of mechanical labours, may well appear strange. We shall have little, therefore, to say upon this subject.

492. *Experiment.* The absolute force of different horses, of mean size and power, having been tried by the dynamometer (472) when they drew horizontally, Regnier thence deduced the mean value, which turns out to be 350 chilograms.

493. Scholium 1. Wherefore, in this kind of exertion, the absolute force of a horse is seven times as great as that of a man. It is not so in the carrying of weights; nor could a horse support a load seven times as great as that of a porter.

494. Scholium 2. With respect to the permanent force of a horse in drawing, Sauveur believes that a horse can work eight hours in a day, exerting a force of 85.66 chilograms, with a velocity of 0.97 metres in a second: whence the effect (475) would be expressed by the number 2392998. The estimate of Smeaton, which reduces this effect to 1716201, is more likely to be true. And, as to the carrying of weights, he computes that a horse can carry 97.9 chilograms through 38980 metres; whence the effect would be 3816142. But I know not how far we may trust to these measures; which, to say the truth, for a labour continued for several days, appear to be too great.

495. Scholium 3. We have, in fact, an observation of Hachette (*Traité des Mach.* p. 12.) from which the permanent force of a horse in drawing, appears to be very much less. He observed, for a long time, the action of a horse, which was

employed in machinery of the following description. The horse, by going round, turns an horizontal wheel, or drum, which, by means of a cord wrapped about it, draws up a bucket of water from a well. The depth of the well is 32.5 meters; the bucket holds 90 pints of water, the weight altogether is 100 chilograms. One bucket full is drawn up in a minute. The duration of the labour varies from one day to another, according to circumstances; but, at a medium, may be taken at five hours. Thus the neat effect will be $100 \times 32.5 \times 300 = 975000$.

496. SCHOLIUM 4. The remarks which we have made, upon the force of man in relation to his velocity, and on the mode of investigating the precise and adequate measure of it, apply also to the force of beasts. And it is to be wished, that the utility of the subject would induce those, who superintend great mechanical works, to direct their attention to this object.

CHAP. XII.

ON FRICTION.

497. To complete our treatise upon active forces, it would remain for us to speak of those, which are derived from the impulse of water, or of wind: but the measure of these belongs to Hydraulics. Proceeding to consider passive and resisting forces, that is, such as are exerted not so much in producing motion, as in hindering it, we shall, for the same reason, omit to speak of the resistance of fluid mediums; restricting ourselves to those which arise from friction, from the rigidity of ropes, and from the tenacity of solids.

498. Three kinds of friction are distinguished from each other, by writers on Mechanics; or rather, friction is considered and measured, in three different kinds of motion. These are, first, that of a body, which slides along a plane; secondly, that of a cylinder, which rolls over a plane; thirdly, that of the axis of a wheel, or of a pulley, whether the axis itself turns, or the wheel turns about the axis.

499. The elements, which can influence friction, are, first, the pressure with which the body weighs on the surface, over which it moves; secondly, the roughness of the surfaces, which rub against each other, depending upon the difference of their nature, and of the state to which they have been brought; thirdly, the duration of their mutual contact; fourthly, their extension; fifthly, the velocity of the motion. Upon the influence of each of these elements, experience alone can afford us any light.

500. Here, however, we cannot place much confidence in experiments made upon a small scale, where the operation of friction may be obscured by that of accidental or extraneous causes, which may happen to mix themselves with it. Now,

two celebrated experimenters, Coulomb (*Mem. Pres. a l'Acad.* tom. X.) and Ximenes (*Terria e Pentica delle Resist. de' Sol. ne' loro Attr. Pisa,* 1782.) by a long series of trials, made upon a large scale, have, contemporaneously, investigated the measure and the varieties, of friction. Their results, for the most part agreeing with each other, throw much light upon this subject: only that Coulomb, having more varied his experiments, has been enabled to point out the remarkable influence of some elements, unobserved by Ximenes; in whose experiments, on this account, there was some irregularity. From these writers, therefore, and especially from the former, we shall borrow our experiments on friction; beginning with that of the first kind.

These experiments are of two sorts. By the first, friction is measured, when the two surfaces rest firmly upon each other, and the intention is to move them from the state of mutual contact. By the second, friction is measured, when the one surface runs over the other with any velocity.

501. *Experiment* I. The tribometer, used for this experiment, by Coulomb, and by Ximenes, is, substantially, the same. Upon a long horizontal wooden table rests a beam, also of wood, which is loaded with different weights. The beam is drawn horizontally, by a flexible cord, which, passing over a pulley, very easily put in motion, has suspended to it a dish, on which greater and greater weights are placed, until a weight is found sufficient to move the beam; and this last weight is the measure of the friction. The quality, and the breadth, of the surfaces subjected to friction, are varied at pleasure; by fastening upon the table, or under the beam, plates of various kinds and sizes. In the following corollaries, we shall relate the results of these experiments.

502. *Coroll.* 1. Under like circumstances, the friction is proportional to the pressure.

If, therefore, Q denote the pressure, the friction may be expressed by $f.Q$, f being an invariable coefficient, whatever be the pressure: except that, from the experiments of Coulomb, and still more clearly from those of Ximenes, it appears that the coefficient f decreases a little, in great pressures. The

difference, however, is small; and may, for the most part, be safely neglected.

503. *Coroll.* 2. The friction varies according to the quality of the surfaces. In new wood, even when it is planed, it amounts to about half the pressure; in metals, to $\frac{1}{4}$; in wood and metals, to $\frac{1}{5}$.

When the surfaces are worn by long attrition, the friction becomes less. Thus, in woods, from being half of the weight, it is reduced to a third. The friction in woods turns out to be much less, when the fibres cross each other at right angles; by which circumstance it is reduced to a fourth of the pressure.

504. *Coroll.* 3. A considerable advantage is obtained by smearing the surfaces with unctuous matter; and the greater consistence the ointment has, the greater is this advantage. Thus an ointment of tallow, when it is fresh, diminishes the friction by one half.

Even when the ointment is not renewed, so long as there remains on the surfaces any small portion of the old unctuosity, the advantage still remains, and we may consider the friction of the unctuous surfaces as a mean between that of dry surfaces, and that of surfaces newly smeared.

505. *Coroll.* 4. The friction varies according to the duration of the contact, and increases for a certain time, until it reaches its greatest and constant value. This time, in woods, is that of a minute or two; in metals, it is very short, and imperceptible; but in woods placed over metals, it lasts several days. It is much prolonged by greasing the surfaces. It is, in general, greater, when the surfaces are extensive, and when they are of different kinds.

506. Scholium 1. On account of this circumstance not having been observed by Ximenes, some irregularity occurred in his experiments. As he did not wait for the greatest augmentation of the friction, he must necessarily have found it to be less than that found by Coulomb. And so, in fact, it was; and the difference shewed itself the most, in all frictions between

woods and metals, and in those cases, in which ointment was used. In the same manner is explained a discrepancy, which is met with, in the results of Ximenes, compared with each other: it being probably owing to the greater or less duration of the preceding contact.

507. *Coroll.* 5. Friction depends little, or not at all, upon the breadth of the surfaces. It is true, indeed, that when the pressure is small and the surface much extended, the resistance is found to be a little greater; especially if a layer of grease interpose: but this very irregular and inconstant increase, probably, arises from cohesion.

508. SCHOLIUM 2. There is another method (Bulfinger, *Com. Petrop.* tom. II.) of investigating the coefficient f of attrition, for a given body. And this consists in finding, by experiment, the steepest plane, on which the body will rest. Let the plane, thus found, decline from the vertical by the angle m; then, $f = \cot. m$. For the body has a tendency to descend with a force $= g . \cos. m$, and it presses the plane with a force $= g . \sin. m$. Wherefore, the friction $= fg . \sin. m$; and since it counterbalances the force with which the body endeavours to descend, we have

$$f . g \sin. m = g . \cos. m; \quad \text{and } f = \cot. m.$$

Thus, it having been found (Perronet, *Mem. de l'Acad.* 1769.) that bricks just rest upon a moderately smooth plane, when it declines 50° from the vertical, for bricks, $f = 0.8$, nearly.

Earth, when put in motion, does not keep its level, but naturally assumes a declivity from the vertical of 60°, if it be sandy and loose, or 54° if it be of closer texture. Wherefore, for the friction in this case, f is from 0.58 to 0.75.

509. *Experiment* II. In this experiment, Coulomb makes use of the same tribometer, except that the dish, which carries the weight, can descend through a long space, drawing the beam through an equal space. He suffers it, in this case, drawn by the weight, to describe the whole length of the table, and, by means of a clock, which shews half seconds, he compares the time, in which the beam describes the first half of the length, with the time, in which it describes the other half.

Sometimes, the first interval of time turns out to be nearly the double of the second. Now, it is easy thence to deduce, that the motion must have been uniformly accelerated. For if $2l$ be put for the length of the table, upon a supposition that the motion was uniformly accelerated, the time, in which the first half, l, is described, will be (210) $t = \frac{\sqrt{2l}}{\sqrt{g}}$, and the time, in which the whole length $2l$ is described will be $t' = \frac{\sqrt{4l}}{\sqrt{g}}$: hence, the time, in which the second half of the length is described, will be $t' - t = \frac{\sqrt{4l} - \sqrt{2l}}{\sqrt{g}}$. Wherefore, the times t, $t' - t$, in which the two halves of the table are successively described, are to one another as $\sqrt{2}$ is to $\sqrt{4} - \sqrt{2}$; which ratio is not very different from that of 2 to 1.

Wherefore, since the first interval of time is found to be nearly the double of the second, we ought to conclude that the motion is equably accelerated, and thence that the friction is constant, and independent of the velocity. From a comparison of the spaces with the times, the value of the moving force might be calculated, and the measure of the friction might thence be deduced.

Upon another occasion, he found the times to be nearly equal, and the motion, as to sense, to be uniform: in which case, we must conclude that the friction increases with the increase of the velocity. The friction, for that degree of velocity which obtained in the experiment, is then measured by the drawing weight.

The following are the results obtained from many trials.

510. *Coroll.* 1. The friction of bodies in motion is, generally speaking, less than that which must be overcome, at their first parting from one another. The variation, however, is not equal in all. In woods, the friction, at the first parting, being the half of the pressure, it is not more, in continued motion, than an eighth part of the pressure. On the contrary, in metals there is no sensible difference between the former and the latter

of these frictions; although between woods and metals, where, as we have seen, the first friction is $\frac{1}{5}$th of the pressure, the second, in a slow and continued motion, is reduced to $\frac{1}{12}$th.

511. *Coroll.* 2. In woods, and much more in metals, the friction appears to be constant, and independent of the velocity. It is not so, where woods and metals rub against each other: here, the friction is sensibly increased with an increase of the velocity, although in a much less proportion.

It is true, nevertheless, that, when the woods have been worn by continued attrition, and especially if the load be heavy, and the surface small, that the velocity no longer much affects the friction.

CHAP. XIII.

ON THE SECOND SPECIES OF FRICTION.

512. *Experiment.* Upon two small parallel tables, placed in a perfectly horizontal plane, and at a short distance asunder, is laid a cylinder, with its axis perpendicular to their length. Over this, in the interval between the two tables, is coiled a pliant string, carrying, at its two extremities, two equal weights, which keep the cylinder in equilibrium. Then one of the two weights is increased so as to produce, and maintain, in the cylinder, a slow and continued motion. This weight is the measure of the friction, which the cylinder has to overcome, when it rolls over the plane.

Here, we may vary, at pleasure, the diameter and the matter of the cylinder. If we wish, also, to vary the pressure, instead of a single string, we may coil about the cylinder two, or more, properly distributed.

And to be more sure of finding the least weight, which breaks the equilibrium, two trials are made every time, by increasing the weight first on the one side, then on the other; and if the difference is small, the mean between the two observed values, is taken. The same precaution is also to be taken, in the experiments, which we shall presently describe.

513. *Coroll.* 1. Under like circumstances, this second species of friction is nearly in an inverse ratio of the diameter of the revolving circle or cylinder.

514. *Coroll.* 2. This second kind of friction also varies according to the different substances, which form the cylinder or the plane. Unctuosity does not seem at all to diminish it.

When a cylinder of mahogany, with a radius of 0.081 metres rolls upon a plane of oak, the experiments give $f = 0.006$; when it rolls upon a plane of elm, it is found that $f = 0.010$. Whence it appears, that the second kind of attrition is very small, compared with the first.

CHAP. XIV.

ON THE THIRD SPECIES OF FRICTION.

515. *Experiment.* Over a pulley, which admits of a rotatory motion, having its axis within a fixt circle, which embraces it, is laid a pliant string, loaded, at its extremities, with equal weights, of magnitudes sometimes greater, sometimes less, in order to vary the pressure. The weight, on one side, is then increased, so as to communicate to the pulley a slow and continued rotation.

To deduce from this additional weight the measure of the friction, we must consider that the weight acts at the extremity of the radius of the pulley, whilst the resistance from the friction is exerted at the extremity of the radius of the axis, that is to say, in the place of contact between the axis and the circle.

We must, therefore, reduce this weight to an equivalent force, which shall be directly opposed to the resistance; for which purpose it is necessary to increase it, in the ratio (107) of the radius of the axis to the radius of the pulley; and the value of the weight, thus augmented, gives the measure of the friction. Attending, then, to the descent of the preponderating weight, and comparing the time, in which it describes the first half, with that in which it describes the other half of the length, it is ascertained (509) whether the friction be constant, or whether it increases with the velocity.

Lastly, it must be observed, that when it is wished greatly to increase the pressure, instead of a string, it becomes necessary to make use of a hempen cord. In this case, the resistance is increased on account of the rigidity of the cord; and the result must be corrected, by subtracting from it the value of this extraneous resistance. How this is to be done, we shall shew in the next following chapter. We proceed now to give the results of Coulomb's experiments.

516. *Coroll.* 1. Under like circumstances, the friction is found to be *nearly* proportional to the pressure ; we say *nearly*, because here, as above (502) under great pressures, the friction is somewhat less iu proportion.

517. *Coroll.* 2. The friction varies according to the materials, of which the axis, and its circle, are formed. If the axis be of iron, aud the circle of brass, the friction is $\frac{1}{7}$th of the pressure ; but when the axis, aud the circle, are of wood, there is a less friction, amounting to about $\frac{1}{12}$th of the pressure.

518. *Coroll.* 3. Here, a great advantage is obtained by greasing the surfaces. Fresh ointment of tallow diminishes the friction one half. It afterwards increases in proportion as the oily matter is consumed ; but this takes place more slowly than it does, in the first kind of friction.

519. SCHOLIUM 1. This result is contradicted by the experiments of Ximenes, who found no sensible advantage from greasing the axles, aud the circles, of the pullies. But it must be remarked, that he made use of thick ropes, without taking account of their rigidity. Now, from the dimensions, which he gives of them, it appears, that this rigidity must have caused a resistance far greater than that which arose from the friction. His results exhibit the mixt effect of the two resistances, but do not supply the data requisite for measuring them separately : nor can we thence deduce any sure inference, with respect to the variation of the attrition, which is the smaller of the two resistances.

520. *Coroll.* 4. The friction of the axles does not at all depend upon the velocity : at least the influence of the velocity is here so little, that, in practice, it may be considered as nothiug.

521. SCHOLIUM 2. It is found that the friction of the axles is less than that of the first kind. Bossut (*Mech. Sect.* 308.) assigns as the reason of it, that we may consider this as a mixt kind of friction, partaking of the first and of the second kind. I know not how far this reason is valid. In the second species of friction the body touches the plane only in a single line, and soon quits it ; but, in the friction of the axles, as well as in that

of the first kind, the two surfaces continually slide over each other, and never quit one another, the only difference between the two cases being that, in the axles, the friction takes place on a cylindrical surface, instead of a plane surface. That which, in my opinion, may diminish the friction of the axles, is rather that, as the surface rubbed is always the same, and as the same track is always pursued, the prominent points, once worn down, cannot cause the resistance which they would do, if it were necessary again to wear them down: whence, during a continuation of the motion, it may well seem that less friction ought to result; as in reality we find it to be.

CHAP. XV.

ON THE RIGIDITY OF ROPES.

522. In order well to understand whence arises the resistance, of which we are now to treat, let us imagine a force applied to raise a weight by drawing a rope, laid over a pulley, or a cylinder. If the rope is not perfectly pliant, it happens, that, whilst it is drawn, it remains, for a short space, separate from the surface of the cylinder; thus increasing the distance of the weight from the axis of rotation, and augmenting its momentum, so that, on this account, a somewhat greater force is required to raise it. This additional force overcomes, and measures the resistance arising from the rigidity. It may vary, first, according to the tension of the rope, or to the weight with which it is loaded; secondly, according to the different quality, construction, and preparation, of the rope; thirdly, according to its size; fourthly, according to the radius of the pulley or cylinder, about which it is coiled. Guided by experiment, we shall investigate the value of each of these elements.

523. *Experiment.* Coulomb tried the rigidity of ropes in two different ways, and obtained results perfectly agreeing with each other. The more simple method is that described in Art. 512; only that, instead of a string, a rope, of greater or less thickness, is made use of. From the weight, which, breaking the equilibrium, produces, and maintains, a slow and continued motion in the cylinder, must be subtracted that part, which overcomes the friction of the second kind; the rest overcomes and measures the resistance of the rope.

524. *Coroll.* 1. The weight, which stretches the rope, varying, and all other circumstances remaining the same, the rigidity of the rope is partly constant, and partly proportional to the tension. So that it may be expressed by $\mu + \nu Q$, Q denoting the stretching weight.

525. *Coroll.* 2. The rigidity varies, and thence the coefficients μ, ν, also vary, according to the different quality, construction, and preparation of the ropes: which embraces a variation almost indefinite, especially, when we consider the many elements that concur to diversify the construction of hempen ropes. Generally speaking, it may be said that ropes are the more rigid, as they are the more new, and the more twisted. Pitched ropes are more rigid than those that are not pitched.

526. *Coroll.* 3. If the radius of the rope, or its circumference, vary, the rigidity varies, no longer in the proportion of the radius, as we first supposed, but in a greater proportion. It varies as a power k of the radius, or of the circumference; the exponent k being greater than unity, but different in ropes of different qualities. In new ropes, and in pitched ropes, we may assume $k = 1.7$; in ropes, which have been long used, $k = 1.4$.

527. *Coroll.* 4. If the radius of the pulley, or of the cylinder, on which the rope is coiled, vary, the rigidity varies inversely as the radius.

528. *Coroll.* 5. The following are the values of the coefficients μ, ν, for two ropes, the one clean, the other smeared with pitch, having a circumference of 0.0632 metres, and wrapt about a cylinder having a radius of 0.0541 metres:

In the clean rope, $\mu = 2.060$ chil.; $\nu = 0.090$ chil.
In the pitched rope, $\mu = 3.230$; $\nu = 0.116$.

Hence we may deduce a value for estimating, nearly, this resistance in every other case.

529. SCHOLIUM. A doubt may arise, as to whether the resistance, which is owing to rigidity, be constant, or whether it increase together with the velocity. In order to remove this doubt, Coulomb observes that if, in the experiment of Art. 515, instead of a string, a rope be made use of, the descent of the preponderant weight will be altogether equably accelerated, and, therefore, the resistance will be constant. Now this resistance has two parts; the one proceeding from friction, the other from the rigidity of the rope. The former of these (520) is constant; wherefore, the latter is also constant.

CHAP. XVI.

ON THE ABSOLUTE RESISTANCE OF SOLIDS.

530. A TENACIOUS cohesion binds together the particles of solid bodies, opposing a strong resistance to any force, which endeavours to separate them. It is manifest that this resistance must vary, not only in different kinds of solids, but also according to their various dimensions, and according to the various modes in which the force is applied to produce a rupture. If all solids were perfectly homogeneous and inflexible, the resistance of every section might be considered as the resultant of so many equal and parallel forces as there are equal elements in the section; whence, it would be proportional to the area of the section, and would be collected in its centre of gravity. In that case, all the varieties, depending upon the dimensions of the solid, and upon the mode of application of the force, might be determined geometrically. Now, although this perfect and equal rigidity of the fibres does never obtain, yet it will be advantageous to assume it, as an hypothesis; and to see, taking Galileo (*Op.* tom. III. p. 66.) for our guide, what consequences follow from it. We shall afterwards have recourse to experiment; and a comparison of the results will teach us, how much ought to be allowed for the parts being heterogeneous, and how much for their imperfect elasticity.

531. Let a prismatic solid (Fig. 39.) be fixed, at one extremity, in a wall, and loaded, at the other extremity, with a weight, which tends to break it in the section $MRSN$; so that there is an equilibrium between the weight, and the resistance of the solid. This weight is taken for the measure of the resistance.

If the weight acts perpendicularly to the section $MRSN$, as does the weight P, tending to pull asunder the solid, the *resistance* is said to be *absolute;* if it acts in a direction parallel to the section $MRSN$, as does the weight Q, tending to break the solid transversely, the *resistance* is said to be *relative.*

532. *Proposition.* If the coefficient k express the tenacity of each element of the section $MRSN$ (fig. 39.) and, this section being referred to the axis AD, about which it is supposed to be symmetrical, if x be the abscissa, and y the ordinate of any point in its perimeter; the absolute resistance of this section, or the weight P, which is in equilibrium with it, will be expressed thus,

$$P = 2\,k\!\int y\,dx.$$

For the elementary trapezium $efgh = 2\,y\,dx$; and its tenacity $= 2\,k\,y\,dx$. Wherefore, the resultant (530) or the whole tenacity of the area $MRSN$, to which the force P is to be equal and opposite, will be $2\,k\!\int y\,dx$.

533. *Coroll.* Wherefore, in homogeneous solids, the absolute resistance is proportional to their sections, made by planes perpendicular to the drawing force.

534. *Experiment* I. Musschenbrock (*Introd. ad Cohær. Corp. Firm.*) tried the absolute resistance of various solids, by the following method. The solids being placed in a vertical position, and their lower extremities being firmly fixt, he tied the other extremity to the shorter arm of a steelyard. Then moving the weight on the longer arm, he thus gradually increased the force, so as to raise up the other arm, and to tear asunder the solid. He noted the place of the moveable weight, at the instant of the rupture, and he thence knew the force required to overcome the tenacity.

He thus made many experiments, especially on different kinds of wood, and upon wires of different metals, and of different thicknesses. Others have since made experiments upon some stones and cements. We shall succinctly give the most remarkable of the results.

535. *Coroll.* 1. In bodies of the same substance, the resistances ought to vary as the transverse sections: now, in metals, they deviate from this law, not very much however, and sometimes in one way, sometimes in the opposite way; and in woods, the aberration was greater, and quite irregular.

It is plain that these anomalies arise from the want of homogeneity; perhaps they would decrease, if the experiments were performed on a large scale; but such experiments are extremely

difficult to perform, on account of the enormous force which they require.

536. *Coroll.* 2. Amongst metals, after gold, that which offers the greatest resistance, is iron; the next are silver and brass, metals of equal tenacity; then follows copper; the weakest are tin and lead. The following is, very nearly, the scale of their tenacities:

Gold....................	1110
Iron....................	1000
Silver..................	820
Brass...................	820
Copper..................	665
Tin.....................	110
Lead....................	65

According to some experiments of Count Rumford, a square centimetre of good beat iron sustains 4470 chilograms; whence we may deduce a rule for the other metals.

By similar experiments, the tenacity of various kinds of stone appears to be of 13.36 chilograms for every square centimetre of surface, and that of bricks is found to be 18.7 chilograms. The resistance of cements was also tried; but this must necessarily be, and was, in fact, found to be, very various. We need not be surprised, therefore, if Coulomb found it to be 3.34 chilograms, whilst Delanges did not find it more than 0.67 chilograms.

537. *Experiment* II. Duhamel (*Art de la Corderie*) with an apparatus not unlike the preceding, investigated the force required to pull asunder thick hempen ropes. Here much variation was to be expected; because, besides the different qualities of the hemp, all the circumstances of the manufacture greatly affect its force. The principal observations were these.

538. *Coroll.* 1. In ropes of the same thread, and manufactured in the same manner, the force was found to be proportional to the section; according to the theory.

But in ropes of different manufacture, no rule can be taken from the sections; because the closer or looser texture of the

threads, or of the collections of threads, causes much variety. The force is found to be more exactly proportional to the weight of the rope, under an equal length; to which weight is proportional the number of the elements, that, by their tenacity, oppose the rupture. This must be understood to be applicable, when the quality of the threads, and the degree of twisting, is the same in both the ropes.

539. *Coroll.* 2. In ropes of good hemp, twisted according to the common practice, that is, so as to be shortened by one third part, from various trials, the absolute resistance was found to be 520 chilograms, for every square centimetre of the section.

But this value is subject to much variation. Above all, it is affected by the degree of twisting, which, in proportion as it is greater, weakens the rope the more. Duhamel found, that, when the ropes were less twisted, so that they remained shortened by a fourth part only, instead of a third, their strength was inincreased in the ratio of two to three. He has also found, that ropes, soaked with pitch, are weaker than those which are clean, and that wetted ropes are weaker than those that are dry.

CHAP. XVII.

ON THE RELATIVE RESISTANCE OF SOLIDS.

540. *Proposition.* The preceding notation being retained, if (Fig. 39.) the length of the solid $AC = c$, the relative resistance of the section $MRSN$ will be expressed by a weight Q, in equilibrium with it, such that

$$Q = \frac{2k}{c} \int xy\,dx.$$

For the weight Q tends to break the solid in the section $MRSN$; making it turn about the lowest boundary RS, and it acts with the momentum $Q \cdot c$. Now, the tenacity of the elementary trapezium $efgh$, which $= 2ky\,dx$, resists this force with a momentum $= 2kxy\,dx$; wherefore, the sum of the momentums, for the whole section will be $= 2k\int xy\,dx$. But, in order that there may be an equilibrium, this sum ought (108) to be equal to the momentum of the weight Q. Wherefore, $Q = \frac{2k}{c}\int xy\,dx$.

541. *Coroll.* 1. Let the solid be of the form of a parallelepiped, as beams usually are; let its depth $AD = a$, its breadth $MN = b$, and its length $AC = c$. Then $Q = \frac{a^2bk}{2c}$; whence, the relative resistances of beams are in a ratio compounded of their breadths, the squares of their depths, and the inverse ratio of their lengths.

542. *Coroll.* 2. The following is the ratio of the absolute to the relative resistance;

$$P : Q :: c\int y\,dx : \int xy\,dx.$$

Wherefore, in beams, the absolute is to the relative resistance, as the length to the half of the depth.

543. *Coroll.* 3. Together with the weight Q, the weight of the solid itself, tends to break it, in the section $MRSN$. If this

weight is to be taken into account, which we shall call V, it must be understood to be applied at the centre of gravity of the solid, that is, at the bisection of its length. It acts, therefore, with the momentum $\frac{1}{2}Vc$. Hence, the equation of equilibrium will be

$$Q+\tfrac{1}{2}V=\frac{2k}{c}\int xy\,dx.$$

544. Scholium I. Quitting Galileo's supposition of the absolute rigidity of the fibres, Leibnitz (*Act. Erud. Lips.* 1684.) proposes another hypothesis, which seems better to suit bodies composed of flexible fibres, capable of being lengthened by stretching. Whilst the weight Q exerts a force, to turn the section about its lowest side RS, the elements, contiguous to this side, are not affected by it; the others are more and more strongly drawn, according to their distance from RS. According to Leibnitz, their resistance is proportional to the separating force which they suffer; whence, if k be put for the resistance at the upper side MN, the resistance at ef will be $=\frac{kx}{a}$. Therefore, the elementary trapezium $efgh$ will resist with the momentum $\frac{2k}{a}.x^2y\,dx$. Equating, as before, the sum of these momentums with the momentum Qc of the weight, according to this hypothesis, we have

$$Q=\frac{2k}{ac}\int x^2y\,dx.$$

Hence, the ratio of the absolute to the relative resistance becomes

$$P\ :\ Q\ ::\ ac\int y\,dx\ :\int x^2y\,dx,$$

that is, in beams, as the length to a third of the depth.

Wherefore, in this hypothesis, the proportion, between the relative resistances of beams, assigned in Art. 541, also obtains.

545. Scholium II. Some writers express the resistance of the ordinate ef simply by kx, and thus find, for parallelepipedal beams, $Q=\frac{a^3bk}{3c}$; so that their resistance would thence vary, no longer in the ratio of the squares, but in that of the cubes, of their depths.

But it is to be observed, that if this resistance, or the separating force at ef, be expressed by kx, the separating force, at the highest side MN, should be expressed by ka; whence, this force would not have a fixt value, but might increase with a, indefinitely; which cannot be admitted. In the state nearest to that of rupture, it is plain, that the greatest separating force, which takes place at MN, ought to be precisely equal to that greater elongation which the fibres of the solid suffer, before they break. And, therefore, the resistance in every point of MN cannot become either greater or less than the absolute resistance, or the tenacity k.

546. SCHOLIUM III. The hypothesis of Leibnitz being admitted, the solid ought to bend a little, before it breaks. The figure, which it assumes, in the state nearest to rupture, is that of an elastic lamina, bent by a weight.

For let $ABCD$, (Fig. 40.) be the profile of the solid, and let it, by the weight Q, be bent into the position $ABdc$. Let the curve Amc be referred to the axis AP, calling the co-ordinates AP, x', and Pm, y', and retaining, in other respects, the former notation. Let the point n be indefinitely nearer to m, and let the sections mh, nk, made perpendicularly to the curve Amc, meet in K. Then $Kn = Km = R$, the radius of curvature of the curve Amc at the point m.

Let us now consider the equilibrium of the weight Q with the tenacity of the section mh, which from the position mi, parallel to nk, is drawn into the position mh, by the stretching of the fibres. Whilst the weight Q tends to make the section mh turn about the point m, removing it continually more and more from the position mi, and thus pulling asunder the already elongated fibres of the solid, every one of these fibres resists, according to Leibnitz, with a force greater in proportion as its distance is greater. Hence, in the point e, the resistance will be proportional to ey; and because $Kn : nm :: me : eq$, if we consider the small arc nm of the curve as constant, eq will be proportional to $\frac{me}{Kn}$, or to $\frac{x}{R}$. Wherefore, if $\frac{h}{R}$ denote the resistance at the point i, where $x = a$, the resistance at the point e will be

equal to $\frac{hx}{aR}$. Hence, the resistance of the elementary trapezium, corresponding to the point e, will be $\frac{hx}{aR} \cdot 2y\,dx$; and its momentum of rotation, about the point m, will be $\frac{hx}{aR} \cdot 2xy\,dx$. Lastly, the sum of the momentums for the whole extent of the section mh, will be $\frac{2h}{aR}\int x^2 y\,dx$.

Let us next observe, that, so long as the solid is supposed to be prismatic, having all its sections equal and similar, the quantity $\frac{2h}{a}\int x^2 y\,dx$ is constant for the whole curve; whence, the definite integral $\int x^2 y\,dx$ is the same for all the sections. Putting, therefore, E for this constant quantity, $\frac{E}{R}$ will be the momentum of the resistance.

On the other hand, the momentum of the weight Q, referred to the point m, about which the rotation tends to take place, is $= Q\,(c-y')$. Wherefore, $\frac{E}{R} = Q\,(c-y')$. But this is exactly the equation (197) of the elastic lamina, bent by a weight.

547. *Experiment.* With small prisms of wood, exactly similar to those, of which he tried the absolute resistance, Musschenbrock investigated the relative resistance; confining one of their extremities in a strong ring of iron, and then loading the other extremity, until the weight was heavy enough to break the prism. He was thus enabled to compare the two resistances; and, varying the dimensions of the prisms, he could also compare together the relative resistances. He submitted to similar trials some small cylinders of glass.

548. *Coroll.* 1. In glass, the proportion of the absolute to the relative resistance, was found to be conformable to Galileo's hypothesis (542); but in woods, it deviated exceedingly, as well from this hypothesis, as from that (544) of Leibnitz; nor did it seem possible to reduce it to any certain law, so various was it in different woods.

This will not occasion surprize to any one, who observes, that the fibres of wood are neither all rigid, as Galileo supposes, nor all equally flexible, as Leibnitz supposes them to be. Hence, no sooner does the weight Q begin to force the solid, than a part of the fibres gives way, whilst the rest resist; and thus the resistance of the section is diminished, in proportion to the number and the position of the broken fibres. On which account, there may be a very great variation.

549. *Coroll.* 2. With respect to the comparison of the relative resistances of solids, that are parallelepipeds, these are found to be as their breadths, and as the squares of their depths, conforming, in this respect, to the theory. How they vary in different lengths was not examined; but the experiments, which we shall describe in the following Chapter, will supply that defect.

CHAP. XVIII.

ON THE RESISTANCE OF SOLIDS SUPPORTED AT BOTH THEIR EXTREMITIES.

550. *Proposition.* If a solid (Fig. 39.), be placed upon two immoveable props, which sustain the extreme sections $MRSN$, $M'R'S'N'$, and if, from the middle point E, there hang a weight T, just heavy enough to be upon the point of breaking the solid, in the section $mrsn$; the resistance of the solid, or the weight T, which is in equilibrium with it, will be expressed thus,

$$T = \frac{8k}{c} \int xy\,dx.$$

For each of the props sustains the half of the weight T; wherefore, the equilibrium will subsist, if, the two props having been removed, we suppose to be substituted for them two forces, each equal to the half of T, which, acting upwards, tend to break the solid in the middle section $mrsn$. The momentum of each of these forces will be $\frac{1}{2} T. \frac{1}{2} c$, or $\frac{1}{4} Tc$. This momentum ought to be equal to the momentum of the resistance, which (540) is $2k\int xy\,dx$. Therefore, $T = \frac{8k}{c}\int xy\,dx$.

551. *Coroll.* 1. Wherefore, $T=4Q$; so that a solid, supported at its two extremities, can sustain four times the weight, which it could sustain, if it were fixt in a wall, at one end only; and this weight, in the case of beams, is as the breadth, and as the square of the depth, directly, and as the length inversely.

552. *Coroll.* 2. If the weight T does not hang from the middle, but from another point, as H, putting $AH=z$, and seeking, in the same manner as before, the equilibrium of the weight with the resistance of the section corresponding to the point H, there will be found

$$T = \frac{2ck}{z(c-z)} \int xy\,dx.$$

Hence, the further the point of suspension H is from the middle, the greater is the weight, which the beam can sustain; this weight varying in the inverse ratio of the rectangle $AH \cdot HC$.

553. *Coroll.* 3. If the solid does not lie horizontally, the weight T must be resolved into two forces, the one perpendicular to the length of the solid, the other in the direction of that length, and account must be taken only of the former.

554. *Coroll.* 4. If the weight V of the beam itself is to be taken into account, it must be considered that each of the props sustains the half of it. Hence, substituting for each of the props an equal force, in a contrary direction, it will be found that this weight acts, on the middle section, with the momentum $\frac{1}{2} Vc$. But, as there is opposed to this force the weight of the half of the beam, included between the sections $MRSN$, $mrsn$, which may be supposed to be applied at the bisection of their distance asunder, and which, therefore, acts with a momentum $= \frac{1}{8} Vc$, the momentum, arising from the weight V, remains equal only to $\frac{1}{8} Vc$. Thus, the equation of equilibrium gives

$$T + \tfrac{1}{2} V = \frac{8k}{c} \int xy\,dx.$$

It will be sufficient, therefore, to add to the load T, the half of the weight of the beam.

555. *Coroll.* 5. Here, an easy problem presents itself.

If there be two horizontal beams, of equal section, the first fixt, at one of its extremities, in a wall, the second supported, at both its extremities, to find the ratio of their lengths, when each of the beams is as long as it can be, so as not to be broken by its own weight.

The solution is readily obtained by means of the formulas of Art. 543, 554, and it is found, that the second may be twice as long as the first. This is, precisely, the conclusion of Galileo, from which De la Hire and Grandi, (*V. Mariano Fontana, Dinamico*, Sect. 328.) erroneously depart.

556. *Experiment.* By great good fortune, we have, on this subject, a beautiful series of experiments, by Buffon, (*Mem. de l'Acad. de Par.* 1740, 1741.) upon thick beams of oak, of a

square section, weighing 1065 chilograms for every cubic metre. These, with their extremities placed on two solid props, he loaded in the middle, by little and little, until he broke them.

557. *Coroll.* 1. The following table exhibits the mean results of the experiments. The weights, to which the beams yielded, are expressed in chilograms; the lengths of the beams, in metres; the sides of the transverse section, in decimetres. To the weight laid on the beam is added the half of the weight of the beam itself, for the reason explained in Art. 554.

Lengths.		Sides of the Square Section.				
		4 in. decim. 1.083	5 in. decim. 1.353	6 in. decim. 1.624	7 in. decim. 1.894	8 in. decim. 2.166.
Feet.	Met.	chil.	chil.	chil.	chil.	chil.
7 =	2.274	2614	5664	9307		
8	2.598	2243	4815	7636	12802	
9	2.923	1988	4095	6478	10996	
10	3.248	1789	3520	5553	9595	13663
12	3.898	1487	3011	4509	7991	11576
14	4.548		2637	3721	6559	9793
16	5.197		2180	3186	5484	8144
18	5.847		1868	2804	4722	6607
20	6.497		1642	2515	4173	5785
22	7.146		1525			
24	7.796		1134			
28	9.095		958			

558. *Coroll.* 2. According to the theory, the weights T ought to follow a ratio compounded of the duplicate ratio of the depths, of the ratio of the breadths, and of the inverse ratio of the lengths. And as here the sections are squares, these weights ought to vary as the cubes of the sides of the sections directly, and as the lengths inversely. From the first proportion they do not deviate much; they deviate from the second more quickly and more constantly; it being evident that the decrease of the weights T is greater than it should be according to the increase of the lengths.

559. *Coroll.* 3. The following might be used as a sufficiently simple formula, which represents the results of the experiments, with an approximation near enough for practical purposes:

$$T = \overset{\text{chil.}}{6180} \cdot \frac{a^3}{c + 0.083\,c^2},$$

where the lengths c are expressed in metres, and the sides a in decimetres.

We might, indeed, render it more exact, by changing its form, or by multiplying its terms; but, the irregularities, which are discovered in the series of results, seem to indicate that it would become somewhat complicate.

560. SCHOLIUM I. Many other particulars, well worthy of attention, were observed by Buffon, in the course of his experiments. For the sake of brevity, we shall point out only two of them. The first is, that the specific gravity often varying even in woods of the same kind, the force was found to be nearly proportional to the specific gravity. The second is an observation, in consequence of which he advises not to load the beams beyond the half of the weight T required to break them; because a weight somewhat less than T, if it cannot soon break them, may bend them sensibly, and in the course of time may break them.

561. SCHOLIUM II. The beam would sustain a load considerably greater, if, instead of being simply placed upon two props, it were immoveably fixt in stone-work, at both its extremities. For, in that case, it cannot break, unless it gives way in three places at the same time. How much the resistance is increased, on this account, has not yet been determined.

In buildings, it is never omitted to enchase the ends of beams in this manner. But, for the most part, no great reliance is to be placed on the advantage of the strength which they thence derive, considering the small resistance, and the ready deterioration, of the cements.

CHAP. XIX.

ON THE RESISTANCE OF SOLIDS TO COMPRESSION.

562. SUPPOSE a prismatic solid to stand vertically on the ground, upon which it firmly rests in AB (Fig. 41.) and to be loaded, at the top, with the weight G. Here, if the parts were perfectly homogeneous and rigid, the action of the weight, pressing the sections of the solid one against another, would aid the tenacity rather than disturb it. To explain, therefore, how it happens, in reality, that columns break under heavy weights, we must recur to that flexibility, of which the fibres of solids, more or less, partake, and recall, upon this occasion, the hypothesis of Leibnitz. Every little defect of homogeneity may then cause the column, giving way in one part or another, to bend, and to verge to a state in which it all but breaks. We shall proceed to investigate, following the steps of Euler, (*Mem. de l'Acad. de Berlin*, 1757.) the conditions of this state.

563. *Proposition.* When the prismatic column $ABDC$ (Fig. 41.) bends under the weight G, placed upon it, the curvature AmC, which, in the state nearest to rupture, it will assume, will be that of an elastic lamina, fixt vertically, and bent by a weight placed on its top.

Let $Cp = x'$, $pm = y'$, the radius of curvature to the curve AmC at m, or $Km, = R$. If we reason concerning the equilibrium, between the weight G and the tenacity at the section mh, as we did in another similar problem (546), we shall find, as before, the momentum of the resistance to be $= \frac{E}{R}$; where E is

equal to the quantity $\frac{2h}{a}\int x^2 y\,dx$, which is constant throughout the whole of the curve. On the other hand, the momentum of the weight is $= G \times pm = Gy'$. Whence we have the equation

$$\frac{E}{R} = Gy',$$

which (199) is exactly the equation of the vertical elastic lamina.

564. *Coroll.* 1. Hence it follows, that, when the inflexion is very small, the equation of the curve $A\,m\,C$, in finite terms, will be (200)

$$y' = f \,.\, \sin.\, x' \,.\, \sqrt{\frac{G}{E}},$$

where the constant quantity f represents the length rs of the greatest swell of the incurvated column, which (202) will fall in the point of bisection of the column.

565. *Coroll.* 2. It also follows, that, in order to have force enough to bend the column by ever so small a quantity, the weight G must, (201) at the least, be equal to $\frac{E\pi^2}{c^2}$. Whence the formula $\frac{E\pi^2}{c^2}$ may be taken as an adequate expression of the resistance of the column.

566. *Coroll.* 3. Let the column be of the form of a parallelepiped, and let the height $AC = c$, let $AB = a$, and let the other dimension, which is not expressed in the figure, be $= b$. Attention must be paid to the difference between these two dimensions a and b, the first of which is posited in the plane, over which the curvature of the column lies, the other is perpendicular to that plane. Now the transverse section of such a column being a rectangle, of the sides a, b, we shall have $\int x^2 y\,dx = \frac{1}{3}\,a^3\,b$, and therefore $E = \frac{1}{3}a^2 b h$. Substituting this value in the formula $\frac{E\pi^2}{c^2}$, which is the general expression for the resistance of

columns, we shall have $\frac{a^2 b h \pi^2}{3 c^2}$; and, hence, the resistance is proportional to the square of the dimension a, to the simple dimension b, and it is inversely as the square of the height c of the column.

If the sections of the columns, which are compared with one another, be similar, as b will then be proportional to a, the resistance will be proportional to $\frac{a^3}{c^2}$. Thus the weight, which a cylindrical column can sustain, will be in a ratio compounded of the cube of the diameter, and of the inverse ratio of the square of the height.

Euler computed the resistance to be proportional to $\frac{a^3 b}{c^2}$, and, when the sections are similar, to $\frac{a^4}{c^2}$: which arises from his expressing the tenacity (*Nov. Act. Petrop.* tom. II. p. 131) in the manner indicated in Art. 545; a manner, which, for the reason there given, does not appear to be admissible.

567. *Experiment.* This is the place to mention a series of experiments made by Musschenbrock, upon small beams of various woods, and of different dimensions, and, also, upon small pilasters of stone, firmly fixt in a table, from which they rise vertically. On the top of each of these he placed an horizontal plane, which he went on loading with weights, until the weight was upon the point of breaking the little beam. He took care that the centre of gravity of the superincumbent weight should fall directly on the axis of the beam, and that it should not change its place whilst the beam was bending.

568. *Coroll.* 1. The weights, which broke the small beams, were found to be exactly in the ratio of $\frac{a^2 b}{c^2}$; which agrees admirably with Euler's theory, modified according to a preceding article.

569. *Coroll.* 2. We shall subjoin some values of weights, sustained by different parallelepipeds, of three decimetres in length, and having a square base with a side of one centimetre.

	chil.
A small beam of oak sustained	131
A small pilaster of bricks	76
A small pilaster of hard stone (*macigno*)	73
One of marble	203

BOOK IV.

ON THE EQUILIBRIUM OF STRUCTURES.

CHAP. I.

GENERAL NOTIONS.

570. THE walls, the columns, the mounds, and other similar masses, which form the bases of the superincumbent or adjacent parts of an edifice, rise vertically from the ground. These bases we shall designate by the generic name of *piers.* The other parts, whilst they thrust against, or press upon, the piers, also mutually act, in like manner, upon each other. And if all these several actions are in equilibrium with one another, and, at the same time, with the resistances, arising from the friction of the surfaces, and from the tenacity of the solid materials and of the cements employed, the edifice is said to be *equipoised.*

571. Every structure is a system of forces, and the examination of its firmness requires the application of the general conditions of equilibrium. But this examination will be facilitated, if we institute it, separately, upon each of the different parts of the structure; provided that we can ascertain and value the forces, which act upon every one of them. It will then be proper, for each separate part, to observe what motions the applied forces tend to impress; and going through these motions, one by one, to make a comparison between the forces, which tend to produce them, and those which resist that tendency.

572. Now, no part of the structure can be dislocated, except it be either by a progressive, or a rotatory motion. For, either this part is displaced, without changing its form, in which case it is as a system of invariable form, incapable of receiving any instantaneous motion, which is not either progressive or rotatory; or else it happens to be displaced, changing at the same time its form: and this, considering the cohesion of tenacity, cannot take place, without the breaking of that part in its weakest section; which generates a progressive motion, if the force acts perpendicularly to the section; and a rotatory motion, if it acts obliquely.

573. We shall distinguish by the name of *stress*, that force which tends to impress a given motion on a given part of the structure; and by the name of *resistance*, that force which tends to hinder it.

Now, in progressive motions it will be necessary to institute a comparison between the *stress*, and the *resistance;* in rotatory motions it will be necessary to compare together the *momentum of the stress*, and the *momentum of the resistance;* which momentums must be referred to that axis about which the system can turn. Upon this comparison depends the conclusion to be drawn, relative to the equilibrium and the stability.

CHAP. II.

ON THE EQUILIBRIUM OF PIERS.

574. In order not to depart from the more simple and usual cases, we shall suppose the pier to be symmetrical about a vertical plane $ABCD$, (Fig. 42.) which passes through the direction of the stress SR. Here, also, will the centre of gravity of the pier lie; and, as all the forces act in that same plane, we need only, instead of the whole solid, consider this profile $ABCD$.

The stress having been resolved into two forces, the one, P, vertical, the other, Q, horizontal, it is clear that the pier cannot altogether give way, unless it be either by a progressive motion, from B towards A, or by a rotatory motion, about the angle A.

575. *Proposition.* To find the resistance of a pier, and the momentum of the resistance.

The progressive motion is resisted by the friction. Let M be the weight of the pier, and P (574) the vertical force of the stress exerted upon it; then the pressure on the base $AB = M + P$, and, therefore, the friction (502), which is the resistance sought, $= f(M + P)$.

The momentum of the resistance (573) must next be sought, in reference to the rotatory motion.

From the centre of gravity G, let there be drawn the perpendicular GX, upon the base, and from a point S, taken at pleasure in the direction of the stress, let there be similarly drawn the perpendicular SE. Let $AX = k$, $AE = x$, $ES = y$.

The horizontal stress Q tends, with a momentum $Q.y$, to overturn the solid, about the angle A. This force is resisted by

the weight M, with a momentum $M.k$, and by the vertical stress P, with a momentum $P.x$. Wherefore, the whole momentum of resistance will be $= Mk + Px$.

576. *Coroll.* 1. In order to the stability of the pier, therefore, we must have $f(M+P) > Q$; and $Mk + Px > Qy$.

If there were an equality between the horizontal stress and the resistance, and between the respective momentums, the pier would certainly stand, but, with every small increase of stress, it would vacillate.

577. *Coroll.* 2. The second condition may be expressed and ascertained, without resolving the stress into the forces P, Q.

From the point A, let fall, on the direction of the stress, the perpendicular AZ. Then, S being put for the whole stress, $M.AX > S.AZ$.

Or, produce the vertical, drawn through the centre of gravity G, until it meet in I, the direction of the stress. Suppose the two forces M and S to be applied at I, and complete the parallelogram, having sides which represent these forces. Then must the diagonal produced, meet the base, on this side of A, towards B.

578. *Coroll.* 3. Let the pier be a rectangular wall; let its height $BC = a$, its thickness $AB = b$, its specific gravity $= G$. Then, $M = abG$, and $k = \frac{1}{2}b$. Wherefore, if the wall sustains only an horizontal pressure Q, the resistance $= abfG$, and the momentum of the resistance $= \frac{1}{2}ab^2G$.

Hence, if the thickness of a rectangular wall be increased, its resistance against an horizontal stress will increase in the ratio of its thickness, and the momentum of the resistance in the duplicate ratio of the thickness.

579. *Coroll.* 4. To shew, by an example, the application of this theory to practice, suppose that a wall, 12 metres in height, and of a specific gravity 2000, is to sustain an horizontal stress of 4500 chilograms exerted at its summit, and let it be required to find the thickness, which it must have so that there shall be an equilibrium.

Here it is necessary to observe, first, that taking the metre for the unit of linear measures, the specific gravity 2000 denotes, that a cubic metre of the material, of which the wall is composed, weighs 2000 chilograms; and thus the product abG will indicate the weight of the wall, for the length of a metre; secondly, that, similarly, when it is said that the value of the horizontal stress is 4500 chilograms, it is the stress, exerted on the unity of length, that is, on the length of a metre, which is meant; thirdly, that the coefficient f of friction may (508) be taken $= 0.8$; for greater security, however, we shall make $f = 0.75$.

Now, in this case, the two equations of equilibrium are

$$abfG = Q; \quad ab^2G = 2\,Qy;$$

where, introducing the numeric values, we shall have, from the first, $b = 0.25$; and from the second, $b = 2.12$.

The wall, therefore, must have the thickness of 25 centimetres, so as not to be displaced horizontally; but, in order not to be overturned, it requires, at least, a thickness of 2.12 metres.

This thickness of 2.12, which is requisite for equilibrium only, will not appear sufficient (576) to secure the stability of the wall. We ought, however, to recollect, that, not having taken into account the tenacity, which binds the wall to its base, we have made its resistance less than it really is; whence, with the computed thickness of 2.12, it may very well sustain a stress greater than 4500 chilograms. Still it will be best, for the sake of abundant caution, somewhat to increase the thickness.

580. Scholium. Hitherto, we have considered the pier as a solid of invariable form, which can only shift its place all in one piece. Since, however, the stress might break it in some one of its sections, it will be proper to investigate its firmness, under this point of view: which we need only refer to, as we have already given the method of estimating the resistance, arising from tenacity.

The vertical stress P tends to crush the pier. It should be compared with the resistance to compression; which is measured as in the nineteenth chapter of the preceding book.

The horizontal stress Q tends to break the structure transversely. And here, considering any section of the foundation, the relative resistance will be found as in the seventeenth chapter of the same book. It is only necessary to observe, that as the weight of that portion of the pier, which leans upon the section, aids its tenacity, by opposing the rotation which the stress Q tends to cause, therefore, instead of the momentum of the stress, the excess of this momentum above the momentum of the weight, which opposes it, must be taken.

CHAP. III.

ON THE SUPPORTS OF PIERS.

581. It is customary to strengthen, externally, rectangular piers, either by means of a sloping wall, which extends through the whole of their length, or by means of buttresses, distributed at intervals. The advantage of these supports is two-fold. They augment the weight of the pier, and thus increase its resistance; and they remove its centre of gravity to a greater distance from the exterior angle, about which it might turn, and thus much more considerably add to the momentum of the resistance. These advantages may easily be subjected to calculation.

582. *Proposition I.* To find the resistance of a wall, with a scarp, against an horizontal stress, and also the momentum of the resistance.

In addition to the notation of Art. 578, and of the preceding articles, let p be the foot or base of the scarp. Then will the products fM, Mk, become

$$afG\,(b+\tfrac{1}{2}p);\quad aG\,(\tfrac{1}{2}b^2+bp+\tfrac{1}{3}p^2).$$

The first of these products expresses the resistance; the second expresses the momentum of the resistance.

583. *Coroll.* 1. For the purpose of mere equilibrium, we must, therefore, have

$$afG\,(b+\tfrac{1}{2}p)=Q;$$

$$aG\,(\tfrac{1}{2}b^2+bp+\tfrac{1}{3}p^2)=Qy.$$

As in the last example, let

$$a=y=12;\quad Q=4500;\quad G=2000:$$

let the breadth of the scarp be a sixth part of the height, so that $p=2$: and let it be required, from these data, to find the thickness b, of the rectangular wall.

From the second equation we shall obtain $b=0.415$. There is no occasion to seek the value of b from the first equation, because, even if $b=0$, the resistance remains superior to the stress.

If it be wished that the thickness of the wall shall be one metre, and if it then be sought what scarp will suffice to preserve its equilibrium, we must make $b=1$, and we shall have $p=1.23$. A scarp, therefore, which is a little more than a tenth part of the height, will be sufficient.

584. *Coroll.* 2. If the same scarp were placed on the inside of the wall, so as to present its slope to the stress, it would be less advantageous; because, then, the momentum of the resistance would only be

$$aG\left(\tfrac{1}{2}b^2+\tfrac{1}{2}bp+\tfrac{1}{6}p^2\right).$$

Still less advantage would result, if instead of adding the scarp, we were to increase the thickness of the whole wall, to the extent of $\frac{1}{2}p$; for the momentum of resistance would then become

$$aG\left(\tfrac{1}{2}b^2+\tfrac{1}{2}bp+\tfrac{1}{8}p^2\right).$$

585. *Proposition II.* To find the resistance of a wall, strengthened externally by equal, and equidistant, parallelepipedal buttresses; and also to find the momentum of the resistance.

Let Fig. 43 represent the ground plan of the structure; and let the two points L, G, bisect the intervals between the buttresses. Supposing the horizontal stress to act equally through the whole length of the wall, it will be sufficient to consider the portion of it exerted between the sections Ll, Gg. And if a vertical plane be drawn through the middle section AB, in this plane will lie both the centre of gravity of the solid, and the direction of the stress tending from B towards A.

Let, now, the wall's height $=a$; its thickness $Ll=BH=b$;

the length of the buttress $HA = c$; its thickness $MN = PQ = p$; the interval $LG = Hh = d$.

Here, the two products $f.M$, $M.k$, become

$$afG\,(bd + cp); \quad aG\,(\tfrac{1}{2}b^2d + bcd + \tfrac{1}{2}c^2p);$$

the first of which gives the resistance, the second the momentum of resistance.

Equating these values with the stress Q, and its momentum Qy, respectively, we shall have, as before, the conditions of equilibrium; and all the dimensions of the wall, or of the buttress, being given, except one, that one may be determined, so as to produce an equilibrium.

586. *Coroll.* 1. Buttresses on the inside of the wall will be found less advantageous; and still less advantageous would it be, to convert the whole solidity of the buttresses into an uniform increase of the thickness of the wall.

587. *Coroll.* 2. Sometimes, the buttress is a prism, having a trapezium for its base, the further side PQ being greater or less than the opposite side MN.

As before, let $MN = p$, and $PQ = q$.

Then, the momentum of the resistance will be found to be, for the external buttress,

$$aG\,\{\tfrac{1}{2}b^2d + bcd + \tfrac{1}{6}c^2(2p + q)\};$$

and for the internal buttresses,

$$aG\,\{\tfrac{1}{2}b^2d + \tfrac{1}{2}bc\,(p+q) + \tfrac{1}{6}c^2\,(p+q)\}.$$

Whence it appears that, for external buttresses, the form $p > q$ is the more advantageous, and the form $p < q$, for internal buttresses.

588. *Coroll.* 3. To shew, by an example, the application of the proposed formulas, let the stress, for the length of a metre, $= 4500$; $G = 2000$; $a = y = 12$; $d = 5$; $c = 3$; $p = 2.3$; $q = 1.6$: and let it be required to find the thickness b of the wall, so that the momentum of resistance may be equal to that

of the stress; the buttress being supposed to be applied externally.

Here, we must observe that, 4500 being put for the stress, on the length of one metre, the stress on the length d, which is 5 metres, will be five times as great, so that $Q = 22500$. And, when the numerical values have been substituted in the equation

$$aG\,\{\tfrac{1}{2}b^2d + bcd + \tfrac{1}{6}c^2(2p+q)\} = Qy,$$

we have $b = 0.127$. Such a thickness, therefore, will render the wall strong enough to sustain the stress.

CHAP. IV.

ON THE STRESS OF THE EARTH OF A TERRACE AGAINST THE WALL, WHICH CONFINES IT.

589. LEMMA. A WEIGHT R having been placed on a plane, inclined to the vertical, at an angle m, to find an horizontal force A, sufficient to sustain it, so that it shall not run down the plane, taking into account the friction.

Each of the forces R, A, having been resolved into two, the one parallel, the other perpendicular, to the plane, there will result parallel to the plane the force $R \cos. m - A \sin. m$; and perpendicular to the plane the force $R \sin. m + A \cos. m$. Now, in order to an equilibrium, the first of these forces ought to be precisely equal to the friction. Wherefore,

$$R \cos. m - A \sin. m = f R \sin. m + f A \cos. m;$$

$$\text{whence } A = R \cdot \frac{1 - f \tan. m}{\tan. m + f}.$$

590. *Coroll.* Hence, if, instead of an horizontal force, the weight R were sustained by a wall, or by any obstacle whatever, the horizontal stress exerted by the weight against the obstacle, would also be

$$R \cdot \frac{1 - f . \tan. m}{\tan. m + f}.$$

591. *Proposition.* To determine the horizontal stress of the terrace $BCEF$ (Fig. 44.) against the wall $ABCD$, and the momentum of this resistance, to overturn the wall about the angle A.

Considering the stress of the triangle of earth BCE, the sloping side of which BE declines from the vertical at the angle $CBE = m$, let PM, pm, parallel to BE, shut in the ele-

mentary trapezium $PpmM$. Let $BC=a$, $CP=x$, $Pp=dx$; then the trapezium $PpmM=xdx.\tan.m$; and, if g denote the specific gravity of the earth, the weight of this trapezium $=gxdx\tan.m$. Wherefore, the horizontal stress, which it exerts against the straight line Pp, will (590) be

$$gxdx\tan.m.\frac{1-f.\tan.m}{\tan.m+f}, \text{ or } gxdx.\frac{1-f.\tan.m}{1+f.\cot.m}.$$

And, if we put $\frac{1-f.\tan.m}{1+f.\cot.m}=M$, this stress will be $Mgxdx$.

Integrating, and afterwards making $x=a$, the whole stress will be $\frac{1}{2}a^2gM$.

Similarly, the momentum of the elementary stress, exerted against Pp, is found to be $Mg(a-x)xdx$; whence, integrating, and then making $x=a$, we shall have the whole momentum of the stress $=\frac{1}{6}a^3gM$.

It remains to fix the angle m. It has been observed, (Coulomb, *Mem. Pres.* &c. tom. VII.) that both the stress, and also its momentum, vanish, whether $\tan.m=0$, or $\tan.m=\frac{1}{f}$. Betwixt these two values, therefore, there is an intermediate value, to which belongs the greatest stress, and the greatest momentum. This value is fouud by making $dM=0$, and is

$$\tan.m=-f+\sqrt{(1+f^2)}.$$

Substituting this value in that of M, the stress is found to be

$$\tfrac{1}{2}a^2g\{-f+\sqrt{(1+f^2)}\}^2=\tfrac{1}{2}a^2g\tan.^2m;$$

and the momentum of the stress is found to be

$$\tfrac{1}{6}a^3g\{-f+\sqrt{(1+f^2)}\}^2=\tfrac{1}{6}a^3g\tan.^2m;$$

which was to be found.

592. *Coroll.* 1. The angle, which has for its tangent $\frac{1}{f}$, is (508) the angle of the slope, which the earth would of itself naturally take, if it were not sustained by any wall.

The angle m, which has for its tangent $-f+\sqrt{(1+f^2)}$, is

precisely the half of the angle, which has for its tangent $\frac{1}{f}$; as may easily be verified, by means of the well known expression for the tangent of the double of an angle.

Let, therefore, BF be the slope, which loose earth would of itself naturally assume. Then, the line BE, which determines the triangle of earth, that exerts the greatest stress against the boundary wall, bisects the angle CBF.

593. *Coroll.* 2. We have seen (508) that sandy and very loose earth, when left to itself, takes a declivity of 60° from the vertical, and that stronger and looser earth takes a declivity of 54°. Wherefore, for a terrace, composed of very loose earth, we have $m = 30^{\circ}$; for another, composed of strong and close earth, $m = 27^{\circ}$.

Hence, for the former kind of terrace, the value of the stress $= \frac{1}{6}a^2 g$, and the momentum of the stress $= \frac{1}{18}a^3 g$. For the latter kind, the stress $= \frac{1}{8}a^2 g$, and its momentum $= \frac{1}{24}a^3 g$.

594. *Coroll.* 3. The horizontal stress of a terrace, and its momentum, being known, it is easy to proportion to them the resistance of the wall $ABCD$, by finding its dimensions, such as are requisite for an equilibrium.

Let the wall $ABCD$ be rectangular, and let it, as well as the terrace, be 12 metres in height: let $G = 2000$, which is commonly the specific gravity of bricks; $g = 1428$, the usual specific gravity of strong earth, of which we will suppose the terrace to be formed: and let it be required to find the thickness b, which the wall must have, so that the momentum of its resistance may be equal to the momentum of the stress.

The momentum of the stress will be found to be $= 102816$; and, equating it with (578) the momentum of resistance, $\frac{1}{2} a b^2 G$, we shall find that $b = 2.93$.

Giving the wall a slope of one-sixth of its height, we shall, in like manner, find by Art. 583, that the thickness $b = 1.15$.

CHAP. V.

ON THE EQUILIBRIUM, AND ON THE STRESS OF POLYGONS.

595. Of several beams, connected together in the form of a polygon, are constructed roofs, wooden bridges, the frames of scaffolding, and other works of a similar kind. The beams, by their own weight, and by the load laid upon them, mutually thrust against, and press upon, each other; and, at the same time, thrust against and press upon the piers, which sustain their extremities. Workmen do not omit to put together these beams, as well as they are able, so as to prevent the shutting or opening of the angles; but as these constructions are neither very strong nor durable, it may be of use to enquire, in the first place, how such a form may be given to the polygon, as that the beams may be in equilibrium amongst themselves. In the second place, we shall have to investigate the stress, which the polygon, thus equipoised, exerts on its piers, in order to proportion thereto their resistance.

For these researches, we have only to recur to Chap. XXI, Book I; where, in Arts. 157, 160, 161, 163, is contained the general solution of the two problems above pointed out, and of many others, which may present themselves, in this subject. Here, we shall content ourselves with indicating the results only, in some of the most obvious cases. We shall suppose that the roof, or polygon of any kind, is symmetrical about the vertical, which passes through its bisection, as well in what regards its form, as in what regards the distribution of the weight. We must also call to mind, that the weights, which load the beams, however they are distributed, may always, (164) be reduced to weights placed on the extremities of the beams, that is, on the angles of the polygon.

596. *Proposition I.* If BCD (Fig. 45.) be a triangular roof, if the weight, placed on the top C, $=2P$, if the weight

applied at each of the points B, D, $= V$, and if the angle $BCK = m$; then, the horizontal stress against the supports B, D, shall $= P \,.\, \tan.\, m$; and the vertical stress shall $= P + V$.

597. *Coroll.* When the breadth BD remains the same, and when the beams are loaded uniformly, and in proportion to their length, whilst the elevation of the roof increases, the horizontal stress, at the extremities B, D, decreases; and the vertical stress increases.

For, then, P and V being proportional to BC, or to $\frac{BK}{\sin.\, m}$, CK being vertical, the horizontal stress will be to the vertical stress, as $\frac{BK}{\cos.\, m}$ is to $\frac{2\,BK}{\sin.\, m}$.

598. *Proposition II.* If $ABDE$ (Fig. 45.) be a quadrangular roof, if the weight at the angle $B = Q$, the weight at the extremity $A = V$, and the angle ABM, which AB makes with the vertical BM, $= n$; then shall the horizontal stress on the supports A, E, $= Q \,.\, \tan.\, n$; and the vertical stress $= Q + V$.

599. *Proposition III.* If $ABCDE$, (Fig. 45.) be a pentagonal roof, if the weight at the top $C = 2\,P$, the weight at the angle $B = Q$, that at the extremity $A = V$, the angle $BCK = m$, and the angle $ABM = n$; then, in order that there may be an equilibrium, must

$$\frac{2\,P}{2 \,.\, \cot.\, m} = \frac{Q}{\cot.\, n - \cot.\, m};$$

$$\text{or} \quad P \,.\, \tan.\, m = (P + Q) \tan.\, n;$$

Also, the horizontal stress, on the supports A, E, shall $= P \,.\, \tan.\, m$; and the vertical stress shall $= P + Q + V$.

600. *Coroll.* 1. If $2\,P = Q$, which happens when the beams are equal and homogeneous, the condition of equilibrium is this,

$$\tan.\, m = 3 \tan.\, n.$$

601. *Coroll.* 2. Here the following problem, (Couplet, *Mem. de l'Acad. de Par.* 1731.) presents itself; "There being given the breadth $AE = 2p$, and the altitude $QC = q$, to form, with four equal beams, an equipoised roof."

Calling the length of a beam a, we shall have these three equations,

$$p = a \sin. n + a \sin. m; \quad q = a \cos. n + a \cos. m; \quad \tan. m = 3 \tan. n;$$

whence may be found the three unknown quantities a, m, n.

But the solution will be more easily obtained thus.

If $AM = x$, $MB = y$, then,

$$\tan. m = \frac{p-x}{q-y}; \quad \tan. n = \frac{x}{y};$$

whence the three equations

$$x^2 + y^2 = a^2; \quad (p-x)^2 + (q-y)^2 = a^2; \quad \frac{p-x}{q-y} = \frac{3x}{y};$$

from which the unknown quantities a, x, y, are easily found.

Thus, if $q = \frac{2}{3}p$, we shall find that $a = 0.618\,p$; and if $q = p$, then $a = 0.732p$.

602. *Proposition IV.* If the pentagonal roof $ABCDE$, (Fig. 45.) is out of equilibrium, and if to prevent the starting of the angles B, D, the points B, D, are connected by a chord, or plank, BD, the weight of which is $= 2R$; then shall the extremities B, D, of the plank, be urged asunder by a force

$$P . \tan. m - (P + Q + R) \tan. n.$$

Also, the supports A, E, will sustain the horizontal stress $(P + Q + R) \tan. n$; and the vertical stress $P + Q + R$.

This will be easily understood, if it be observed, that, in this manner, a system is formed, consisting of the triangular roof BCD, and the quadrangular roof $ABDE$, compounded together. The first exerts (596) on the points B, D, the horizontal stress $= P . \tan. m$, outwards; and the vertical stress $= P$. Hence, the trapezium $ABDE$ is loaded, at the angles B, D, with the weight $P + Q + R$; and therefore, (598) it exerts on the supports A, E, the horizontal stress $= (P + Q + R) \tan. n$, outwards; and (160) the points B, D, are pressed inwards with an equal force.

603. Scholium I. Hitherto, we have considered the beams, as if they were plates of inseparable firmness. But, in reality, they are subject to the possibility of breaking, in conse-

quence of the weight placed upon them, or of their own weight; they may also yield to the compression, or to the stress, which they suffer in the direction of their length, and the value of which (161) it is easy to determine. Now, if the forces be known, to the action of which each beam is subject, we may easily, by means of Chap. XVIII and XIX, Book III, examine their resistance, and find the dimensions, which they ought to have, in order that they may resist as much as is necessary.

604. SCHOLIUM II. Whenever a plank BD, or AE, connects the two supports, it exempts the piers from the necessity of resisting the horizontal stress. For, then, the points B, D, or A, E, cannot move outwards, without breaking the plank itself; against which rupture it opposes, most commonly, a more than sufficient resistance. It will, however, always be proper, by means of Chap. XVI, Book III, to ascertain that the plank resists at much as is required.

605. SCHOLIUM III. But since this plank might break by its own weight, it is usual to connect it with one or more of the angles of the polygon, by means of upright beams, or small pillars. Thus, the plank BD is tied to the triangular roof BCD, by the beam CK; and thus, also, the plank AE may be sustained by the beams BM, DN. It remains for us to enquire what stresses arise from this construction.

The beam BD tends (554) to break in its middle point K, as if the half of its own weight were applied at that point. Hence, the small pillar CK must act with a force equivalent to the half of the weight of BD; so that the ridge C is loaded with the weight of CK, together with half of the weight of BD. This being the case, it will be easy to find the increase of the resulting horizontal stress, against the extremities B, D.

Similarly, the beam AE tends to break in M, with a force (552) so much less than the half of its weight, as the rectangle $AM.ME$ is less than the square of AK. With this force, then, besides the weight of the pillars BM, DN, must the angles B, D, be supposed to be loaded.

CHAP. VI.

ON THE EQUILIBRIUM OF ARCHES AND OF DOMES.

606. In order to treat methodically of this most important part of Statical Architecture, it may be well, in the first place, to call to mind what has been demonstrated in our first Book, relative to the equilibrium of arches, supposed to be constructed of rigid and heavy small sides, simply resting against each other; and so, likewise, of domes, formed in a similar manner, of an infinite number of heavy planes or faces.

607. The curve of equilibrium of an arch, or of a vault, composed of very small rigid sides, that are heavy, or that are loaded, is (189) a catenary; and, calling Π the weight of the arch, it has for its equation $\Pi dy = A dx$.

608. Hence, if all the sides are equally heavy, or equally loaded, the figure of the equipoised arch will be that of an homogeneous catenary, and may be constructed by the determination of points, or mechanically, as (183, 184) we have taught in its proper place. But if the weight is not equably distributed, the curve will be another catenary, the figure and construction of which will depend upon the law, according to which the weight is distributed through the length of the arch; that is, upon the function which Π will be of the co-ordinates of the curve.

609. The equipoised surface of a dome constructed of very small circular zones, heavy or loaded, arises (189) from the revolution of a curve, which, calling $y\,d\Pi$ the weight of the elementary zone, has for its equation $dy\int y\,d\Pi = A dx$.

610. If the faces, composing the dome, are all equally heavy, or equally loaded, $d\Pi$ will be proportional to ds, and the

curve will have for its equation $dy \int y ds = A dx$; or, because $ds = dy \sqrt{\left(1 + \frac{dx^2}{dy^2}\right)}$,

$$y dy = \frac{A d \frac{dx}{dy}}{\sqrt{\left(1 + \frac{dx^2}{dy^2}\right)}}.$$

611. Integrating in a series, we may find (Bouguer, *Mem. de l'Acad. des. Sc. de. Par.* 1734.) a method of describing, through points, the curve which, by its revolution, gives the surface of the homogeneous equipoised dome, and which also gives that of an homogeneous sheet, hanging from the circumference of an horizontal circle.

Let $\frac{dx}{dy} = z$, and we shall have the equation

$$y dy = \frac{A dz}{\sqrt{(1 + z^2)}};$$

whence, integrating by a series,

$$y^2 = 2Az - \frac{Az^3}{3} + \frac{3Az^5}{4.5} - \frac{3.5.Az^7}{4.6.7} + \&c.$$

and, by the reversion of the series,

$$z = \frac{dx}{dy} = \frac{y^2}{2A} + \frac{y^6}{2.4.6.A^3} + \frac{y^{10}}{2.4.6.8.10.A^5} + \&c.$$

Multiplying by dy, and integrating again, we shall finally have

$$x = \frac{y^3}{2.3.A} + \frac{y^7}{2.4.6.7.A^3} + \frac{y^{11}}{2.4.6.8.10.11.A^5}$$
$$+ \frac{y^{15}}{2.4.6.8.10.12.14.15.A^7} + \&c.$$

where the law of continuation is manifest.

612. The constant quantity A will be determined, whenever the ordinate is given, which corresponds to a given abscissa. Supposing, that to the ordinate $x = p$, the corresponding ordinate $y = q$, and making $A = \frac{q^3}{m}$, then

$$p = \frac{m}{6} + \frac{m^3}{336q^2} + \frac{m^5}{42240q^4} + \frac{m^7}{9676800q^6} + \&c.;$$

whence, by the reversion of the series, we shall have

$$m = 6p - 3.857\frac{p^3}{q^2} + 6.334\frac{p^5}{q^4} - 13.622\frac{p^7}{q^6} + \&c.;$$

and thence A is determined.

Thus, if to the abscissa $x = 1$, the corresponding ordinate is $y = 10$, we shall have, with a very few terms of the series, $m = 5.96188$; whence $A = 167.73$.

613. Lastly, it is necessary to recollect, that the equation $dy\int y\,d\Pi = A\,dx$, gives the figure of a dome, which will be equipoised, not only when it is shut in all around, by circular zones, entire and returning into themselves, but also when it is open and broken; so that an ungula of it, insulated and resting, at its top, only against the opposite ungula, will maintain itself in a state of equilibrium: and if the dome is entire, it will not be necessary, for the formation of its equipoised surface, to take the curve defined by the equation $dy\int y\,d\Pi = A\,dx$, but (189) for that purpose any curve may serve, in which $dy\int y\,d\Pi > A\,dx$, or $\frac{dy}{dx} > \frac{A}{\int y\,d\Pi}$.

614. The curve, therefore, of the equation $dy\int y\,d\Pi = A\,dx$, having been traced, every other curve, which, springing from the same vertex, deviates more than that from the axis, and is throughout less concave towards the same axis, will serve, by its revolution, to describe the surface of an equipoised dome.

615. These principles having been recalled, we shall now pass on to consider arches and domes, as they are in reality, composed, not of lines or of planes, but actually of wedges of finite thickness, resting against one another, and supporting each other by their mutual stress.

CHAP. VII.

ON ARCHES OF FINITE THICKNESS.

616. LET the arch KaK' (Fig. 46), symmetrical about the vertical axis B, be formed of an infinite number of heavy wedges $MmnN$, contiguous, but not fastened to one another, and let it be supported on the immoveable beds Kk, $K'k'$. Taking any portion of the arch, beginning from the vertex, as $AamM$, which rests on the inclined bed Mm, it is manifest that this arch is drawn downwards by its own weight, and at the same time is pushed horizontally by the pressure of the opposite arch $K'Aak'$, exerted on the line Aa. Since, therefore, this arch rests on the basis Mm, it is necessary (Coulomb, *Mem. Pres.* &c. tom. VII.) for a state of equilibrium, that the resultant of the two acting forces shall be perpendicular to Mm, and that it shall not fall without Mm. If the first condition be wanting, the arch will be displaced, sliding along Mm; and if the second be wanting, it will have a rotatory motion about that extremity, M or m, towards which the resultant falls. We are to consider, then, what figure the arch must have, in order to satisfy these two conditions.

617. *Proposition I.* The equilibrium of the arch KaK' (Fig. 46.) requires, first, that if any portion of it, as $AamM$, resting on the bed Mm, be taken, the weight of this portion shall be as the tangent of the angle, which the bed Mm makes with the vertical line.

Let Π be the weight of the arch $AamM$, A the horizontal stress on Aa, e the inclination of Mm to the vertical line. Let the two forces Π, A, expressed by the vertical straight line GV and the horizontal straight line GO, meet in G. Wherefore,

since the resultant GS must be perpendicular to Mm, the triangle of forces GSO will be similar to the right-angled triangle MmR, made upon the hypotenuse Mm; MR being drawn perpendicular to the horizontal straight line Vm. Hence,

$$\begin{aligned} GV : GO &:: mR : MR, \\ \text{or, } \Pi : A &:: \sin. e : \cos. e \\ &:: \tan. e : 1. \end{aligned}$$

wherefore $\Pi = A.\ \tan. e$; and Π is varied as $\tan. e$.

618. *Coroll.* 1. An arch cannot be sustained by its own weight, when it is placed on horizontal beds.

For e cannot become 90°, without the $\tan. e$, and therefore also Π, becoming infinite; which is impossible.

619. *Coroll.* 2. Let the inner curve AMK be referred to the vertical axis AB; let $AP = x$, $PM = y$, $AM = s$, and therefore $MN = ds$; also, let z be the thickness Mm, of the wedge, which answers to the abscissa x; and let it be required to calculate the area of the quadrilateral figure $MmnN$, which represents the profile of the wedge, and consequently also its weight $d\,\Pi$.

Draw Np parallel to Mm. In the triangle nNp, we have the angle $nNp = de$; and if from the centre N, at the distance Np there be described a circle cutting Nn in q, then $pq = Np\,.\,de = z\,de$; because, the points M, N, being indefinitely near to each other, we may assume $Np = Nn = Mm = z$, the differences of these straight lines being infinitely small. Hence the area of the triangle nNp will be $= \frac{1}{2} Nn\,.\,pq = \frac{1}{2} z^2.\,de$.

In like manner, the difference of the parallels Mm, Np, being thus infinitely small, the trapezium $MmpN$ may be considered as a parallelogram, of which the area will be $= Mm\,.\,MN.\sin. mMN$. But the angle mMN is the supplement of the sum of the angles e, NMr, and therefore

$$MN.\sin. mMN = MN.\,(\sin. e\,\cos. NMr + \cos. e\,\sin. NMr)$$
$$= dx\,\sin. e + dy\,\cos. e;$$

wherefore, the trapezium $MmpN = z\,(dx\,\sin. e + dy\,\cos. e.)$

Adding together the areas of the triangle and of the trapezium, we shall have, for the area of the quadrilateral,

$$MmnN = d\Pi = \tfrac{1}{2} z^2 de + z\,(dx \text{ sin. } e + dy \text{ cos. } e).$$

But the equation (167) $\Pi = A.$ tan. e, differentiated, gives $d\Pi = \dfrac{A\,de}{\text{cos.}^2\, e}$. Wherefore,

$$\tfrac{1}{2} z^2 de + z\,(dx \text{ sin. } e + dy \text{ cos. } e) = \frac{A\,de}{\text{cos.}^2\, e}.$$

By means of this equation, if there be given the inner curve AK, and the law, which the obliquities of the wedges follow, we can assign the thickness Q proper for each point of the arch; or, *vice versâ* if the thicknesses be given, we can find the directions of the edges, to be given to the wedges, so that the arch may be equipoised.

620. *Coroll.* 3. Let us consider specially the case, in which the edges of the wedges are perpendicular to the inner curve, which case is the most frequent in practice, and offers the most remarkable particulars. Here, the angle mMN being a right angle, the angle e will be the complement of the angle NMr. Hence, tan. $e = \dfrac{dx}{dy}$; and the equation $\Pi = A.$ tan. e, becomes $\Pi\,dy = A\,dx$, the equation of the catenary.

Wherefore, when the wedges press perpendicularly on the inner curve, this curve ought to be a catenary, just as (607) in the case of linear arches. If the arch is homogeneous, and of uniform thickness, it ought to have the form of the homogeneous catenary; if not, the figure of the catenary will be determined by the law, according to which the weight is distributed along the back of the arch.

621. *Coroll.* 4. To find the thickness z, belonging to each point of the arch, in the case of Coroll. 3, we must have recourse to the equation (619) between z and e; which, in this particular case, is reduced to

$$\tfrac{1}{2} z^2 de + z\,ds = \frac{A\,de}{\text{cos.}^2\, e}.$$

Let R be the radius of curvature, at the point M, of the curve AMK; then $R = \frac{ds}{de}$; whence, the equation becomes

$$z^2 + 2\,Rz = \frac{2\,A}{\cos.^2 e};$$

$$\text{whence, } z = R + \sqrt{\left(R^2 + \frac{2\,A}{\cos.^2 e}\right)}.$$

The constant quantity A is easily determined, when the thickness of the vault is known at a given point; for example, at the keystone Aa, where the angle $e = 0$.

622. *Coroll.* 5. Let AMK be a circular arch; let $Aa = m$; and let it be required to determine the thickness of the vault, at a distance of 45° from the vertex.

Here, R will be constant and equal to the radius of the arc; and, since $z = m$, when $e = 0$, we shall have

$$m^2 + 2\,Rm = 2\,A,$$

$$\text{and } z = -\,R + \sqrt{\left(R^2 + \frac{2\,Rm + m^2}{\cos.^2 e}\right)}.$$

Now, when $e = 45^0$, $\cos.^2 e = \frac{1}{2}$; whence

$$z = -\,R + \sqrt{(R^2 + 4\,Rm + 2\,m^2)}.$$

This value, when R is much greater than m, differs very little from $2\,m$. Hence, at the distance of 45^0, the thickness ought to be little less than the double of its thickness at the keystone.

623. *Proposition II.* The equilibrium of the arch KaK' (Fig. 46.) requires, in the second place, that, any portion of it $AamM$ (Fig. 47), beginning from the vertex, having been taken, the vertical line, drawn through the centre of gravity of this portion, shall cross the parallelogram $fghi$, contained by the perpendiculars drawn at the extremities of the boundaries Aa, Mm.

For, otherwise, the resultant of the two forces Π, A, will fall without the basis Mm, on which the arch rests.

It is, besides, manifest, that, if the vertical, drawn through the centre of gravity of the arch, fall without the parallelogram

fghi, from no point of that vertical can there be drawn two perpendiculars, the one to *Aa*, the other to *Mm*; and, therefore, the weight of the arch *AamM* cannot be sustained by the two supports *Aa*, *Mm*.

624. SCHOLIUM. When the sides *Aa*, *Mm*, are perpendicular to the inner curve, we can easily assure ourselves that, whilst the first condition of equilibrium is fulfilled, this second is also fulfilled. For, the inner curve being then (620) a catenary, if he weight of each wedge were wholly collected in the corresponding small side *MN*, the vertical, drawn through the centre of gravity of the arch *AamM*, would pass (162) through the point *f*, of the intersection of the extreme tangents *Af*, *Mf*. Through *m* let there now be drawn the curve *mC*, perpendicular to the produced sides of the wedges, and let this curve meet the edge *Aa*, of the vertex, in *C*. It will also be a catenary; and if the weight of every wedge were collected in the corresponding indefinitely small side of the curve *mC*, the vertical, drawn through the centre of gravity of the arch *AamM*, would pass through *p* the point of intersection of the extreme tangents *Cp*, *mp*.

Wherefore the weight of the arch being, in reality, distributed between the two curves *AM*, *Cm*, the vertical drawn from its centre of gravity will fall between the points *f*, *p*. And here it is easily seen, that, since $AC = Mm$, the parallelogram *fgpq* will be a rhombus, having its obtuse angles in *f*, *p*; and therefore, this vertical falling between *f* and *p*, cannot but cross the parallelogram *fghi*.

CHAP. VIII.

ON PLANE, OR FLAT-SIDED VAULTS.

625. Let KaK' (Fig. 48.) be an horizontal flat-sided vault, of uniform thickness, resting on the supports Kk, $K'k'$, and composed of an infinite number of wedges, as $MmnN$, equally disposed on each side of the vertical key-stone Aa. Applying, to this kind of vaults, the two conditions of equilibrium (616) above laid down, let us enquire how they ought to be constructed, so that they may be equipoised.

626. *Proposition I.* The equilibrium of a flat vault KaK' (Fig. 48.) requires, first, that the tangent of inclination of each side Mm to the vertical, shall be proportional to the distance AM, of the origin of that side from the key-stone Aa.

For, retaining and applying the former notation, we must have $\Pi = A \cdot \tan. e$.

$$\text{But } \Pi = AamM = Aa \cdot AM + \tfrac{1}{2}\,\overline{Aa}^2 . \tan. e.$$

$$\text{wherefore, } Aa \cdot AM + \tfrac{1}{2}\,\overline{Aa}^2 . \tan. e = A \cdot \tan. e\,;$$

$$\text{and } Aa \cdot AM = (A - \tfrac{1}{2}\,\overline{Aa}^2) \tan. e,$$

where, A and Aa being constant, it is manifest that AM is proportional to the tangent of e.

627. *Coroll.* Hence it follows that the sides, produced, of the wedges must all meet in the same point C, of the vertical Aa.

For let C be the point in which the basis Mm meets the vertical. The angle $ACM = e$, and $AC = \dfrac{AM}{\tan. e}$. But (626) AM is proportional to the tan. e; wherefore, AC is constant throughout the flat vault.

The position, therefore, of the support Kk having been settled, and thence also the centre C, where the line of that support meets Aa, if the breadth KK' be divided into any number of parts, the straight lines, drawn from the centre C, to all the points of division, will mark the directions to be given to the sides of the wedges, so as to construct from them an equipoised flat vault.

628. *Proposition II.* The equilibrium of a flat vault KaK' (Fig. 48.) requires, in the second place, that a straight line KP having been drawn, from the origin K of the support, perpendicular to Kk, the vertical, drawn through the centre of gravity of the half of the vault $AakK$, shall cross the triangle KPk.

This follows from Art. 623. And, as, proceeding from the support towards the keystone, the sides Mm are always less and less oblique to the vertical (626), it is easy to assure ourselves that, if this condition of equilibrium be fulfilled for the whole of the semi-vault $AakK$, much more will it be fulfilled for any portion $AamM$ of the vault.

629. *Coroll.* 1. In order to express, analytically, this condition, let us observe that it is reduced to this, namely, that the distance of the centre of gravity of the semi-vault, from the keystone Aa, must be greater than aP.

Let, then, the length $AK = a$, the thickness $Aa = m$, and let Qk, which determines the inclination of the support, $= x$. Computing the distance of the centre of gravity of the trapezium $AakK$ from the side Aa, we shall find it to be

$$\tfrac{1}{2}a + \tfrac{1}{6}\cdot\frac{3ax + 2x^2}{2a + x}.$$

On the other hand, it is easily found that $aP = \dfrac{ax - m^2}{x}$.

Wherefore, we must have

$$\tfrac{1}{2}a + \tfrac{1}{6}\cdot\frac{3ax + 2x^2}{2a + x} > \frac{ax - m^2}{x};$$

$$\text{or, } x^3 - 3(a^2 - m^2)x + 6am^2 = 0.$$

Thus, two of the three elements a, m, x, being given, we can determine the limit of the third.

630. *Coroll.* 2. For an example, let us suppose a flat vault constructed upon an equilateral triangle, so that the support declines from the vertical by an angle of 30^0; and let it be required to find how far its span, or breadth, KK' may be extended.

Let the thickness $Aa = m = 1$; and, since the angle $QKk = 30$, $Qk = x = \tan. 30 = \frac{1}{\sqrt{3}}$. Substituting these values we shall find $a < 3.76$; whence $2a = KK' < 7.52$. Wherefore, the span can be, at most, six times and a half greater than the thickness.

CHAP. IX.

ON DOMES OF FINITE THICKNESS.

631. Let Figure 49 represent the solid ungula of a symmetrical dome, generated by the revolution of the curve AMK, and let $K\lambda'$ be the corresponding portion of the immoveable support, on which the dome rests. In order that the insulated ungula, sustained only by the equal and opposite ungula, shall be in equilibrium, regard must be had to conditions, analogous to those which determine the equilibrium of arches. Any portion $Ma\mu'$, which originates at the vertex, having been taken, the resultant of its weight, and of the horizontal force exerted against Aa, by the opposite ungula, ought to be perpendicular to the bed $M\mu'$, and to pass through some point of that bed. Considering, however, the equilibrium, not of the single ungula, but of the whole dome, that resultant may decline from the perpendicular let fall upon the bed $M\mu'$, provided that it declines from it leaning towards the axis, or towards the interior of the dome, and not towards the opposite part. For if the resultant decline, leaning towards the axis, the ungula $Ma\mu'$ will tend to slip downwards from m towards M; but, there being the same tendency in all the other ungulas, which are about it, throughout the whole round of the dome, and this tendency being exerted by them all at the same time, and with equal force, it is evident that these forces counteract and destroy each other, and that their effect cannot be that the dome shall fall in pieces. If, on the contrary, the resultant lean outwards, the ungula will tend to slide from M towards m, pushing the dome outwards, to which force there is no opposition. Hence arises the following fundamental condition of equilibrium.

632. *Proposition I.* The equilibrium of a dome requires, that, any portions of an ungula having been taken from the vertex, as $Ma\mu$, $Na\nu'$, &c. (Fig. 49.) their weights shall increase in a greater ratio than that of the tangents of the obliquities of the planes $M\mu'$, $N\nu'$, &c. to the vertical line.

Resuming the construction and the figure of Art. 617, since here GS (Fig. 46.) may be oblique to the bed Mm, provided that it falls towards the axis AB, we shall have, generally, $\frac{GV}{GO} > \frac{mR}{MR}$. Calling, therefore, $\int y\,d\Pi$ the weight of the ungula $Ma\mu'$ (Fig. 49.), A the horizontal stress at the vertex, e the inclination of the plane $M\mu'$ to the vertical, we shall have $\int \frac{y\,d\Pi}{A} > \frac{\text{sin. } e}{\text{cos. } e}$, or $\int y\,d\Pi > A \,.\, \text{tan. } e$. And here, since the quantities $\int y\,d\Pi$, and tan. e, both of them begin from nothing, and go on increasing continually, the first set of quantities must needs increase in a greater ratio than the second; whence, differentiating, $y\,d\Pi > \frac{A\,de}{\text{cos.}^2\, e}$.

633. *Coroll.* 1. No dome can remain equipoised by itself, when it rests upon an horizontal support.

634. *Coroll.* 2. Let z denote the thickness Mm of the dome at the point M, to which x is the corresponding abscissa; and let it be required to compute the solidity of $M\nu'$, which expresses also the weight $y\,d\Pi$ of the wedge. This solidity will be found (83) by multiplying the area of the quadrilateral $MmnN$, by the path described by the centre of gravity G. The centre G is in the right line FH, which bisects the opposite sides MN, mn; and since $MN = ds$, and $mn = ds + z\,de$, therefore (76) $FG = \frac{1}{3}z \,.\, \frac{3ds + 2z\,de}{2ds + z\,dc}$. Hence, we have $GQ = FG \,.\, \text{sin. } e$; and the radius of the circle, described by G, $= y + GQ = y + FG \,.\, \text{sin. } e$. Expressing, therefore, by unity the value of the angle subtended by the arc $M\mu$, the path of the centre of gravity

$$= y + \tfrac{1}{3} z \text{ sin. } e \,.\, \frac{3ds + 2z\,de}{2ds + z\,de}.$$

Lastly, multiplying this path by the area of the quadrilateral $MmnN$, already (619) calculated, we shall have

$$M\nu' = y\,d\Pi = \{\tfrac{1}{2} z^2 de + z\,(dx \text{ sin. } e + dy \text{ cos. } e\}$$
$$\times \left\{ y + \tfrac{1}{3} z \text{ sin. } e \, \frac{3ds + 2z\,de}{2ds + z\,de} \right\}.$$

And the thickness must, in every point be such, that this expression never becomes less than $\frac{A\,de}{\cos.^2 e}$.

635. *Coroll.* 3. If we confine ourselves to the case of sections perpendicular to the inner curve, which is so frequent in practice, that it may be considered as universal, the expression for the solidity of $M\nu'$ then becomes much more simple.

For since $\sin. e = \frac{dx}{ds}$, and $\cos. e = \frac{dy}{ds}$, and $\tan. e = \frac{dx}{dy}$, and the radius of curvature $R = \frac{ds}{de}$, we shall find

$$M\nu' = (\tfrac{1}{2}z^2 + Rz) \cdot \left(y + \tfrac{1}{3}z \sin. e \frac{3R + 2z}{2R + z}\right) \cdot de,$$

and if the thickness z be, in every point, sufficiently small in comparison of the radius R, we shall have, in still more simple terms,

$$M\nu' = zyds + \tfrac{1}{2}z^2dx.$$

Lastly, let it be observed, that when y is sufficiently small in comparison of z, the element dx is commonly very small, compared with the element ds, and the second term of the above value of $M\nu'$ may be neglected, in comparison of the first term. And much more may this be done, when y has increased, so as to have become much greater than z. Hence, it should seem, that neglecting the second term, we might make $M\nu' = yzds$, as Bouguer, (*Mem. de l'Acad. des Sci. de Par.* 1734.) has done, notwithstanding the objections of Mascheroni, (*Nuove Richerche sull' equ. delle Volte*, p. 89.) against it.

636. *Coroll.* 4. Making, therefore, $M\nu' = yd\Pi = yzds$, we shall have (632) for the condition of equilibrium,

$$\int yd\Pi > A \cdot \tan.^{\cdot} e, \text{ or } \int yzds > \frac{A\,dx}{dy}.$$

Hence, taking the logarithms, there will result

$$\log. \int yzds > \log. \frac{A\,dx}{dy};$$

and, differentiating,

$$\frac{yzds}{\int yzds} > \frac{dy}{dx} d . \frac{dx}{dy}.$$

By means of this comparison, when the curve is given, which, by its revolution, describes the concave surface of the dome, we shall be able to regulate the thicknesses of the wedges, in their respective places, so that the equilibrium may be preserved; or, the curve itself being given, and also the thicknesses, we can judge whether a dome be equipoised; the method of doing which, we shall explain in the two following Propositions.

637. *Proposition II.* The generating curve of a dome being given, to find the scale of the thicknesses requisite for the state of equilibrium.

It being necessary that $\dfrac{yzds}{\int yzds} > \dfrac{dy}{dx} d . \dfrac{dx}{dy}$,

$$\text{let } \frac{yzds}{\int yzds} = \frac{pdy}{dx} d . \frac{dx}{dy},$$

p being a positive coefficient, and greater than unity. Integrating, we shall have

$$\int yzds = A \left(\frac{dx}{dy}\right)^p;$$

$$\text{whence, } z = \frac{Ap}{yds} \left(\frac{dx}{dy}\right)^{p-1} \times d . \frac{dx}{dy}.$$

638. *Coroll.* Let the dome, for example, be spherical, and let R be the radius of the sphere.

The equation of the generating circle will be $y^2 = 2Rx - x^2$;

$$\text{also, } yds = Rdx; \ \frac{dx}{dy} = \frac{y}{R-x} = \frac{\sqrt{2Rx - x^2}}{R-x}.$$

Substituting these values, we have

$$z = \frac{pAR\,(2Rx - x^2)^{\frac{1}{2}p-1}}{(R-x)^{p+1}},$$

and for p we may take any positive number greater than unity. If the thickness m, which the dome ought to have at its key-stone Aa, be given, when $x = 0$, we must make $z = m$. It is

convenient therefore to assume $p=2$; $A=\frac{1}{2}mR$; and we shall have for the scale of thicknesses,

$$z=\frac{mR^3}{(R-x)^3}.$$

639. *Proposition III.* The generating curve of a dome being given, and also the scale of thicknesses, to find whether the dome will satisfy the conditions of equilibrium.

Here, z and y being known functions of x, it will suffice to observe if the condition $\frac{yzds}{\int yzds} > \frac{dy}{dx} d . \frac{dx}{dy}$ be verified for any value whatever of x.

640. *Coroll.* Again, let the dome be spherical; and let it be required to find if it can sustain itself, having a constant thickness $= m$.

The condition of equilibrium becomes

$$\frac{dx}{x} > \frac{R^2dx}{(R-x)(2Rx-x^2)},$$

$$\text{or } (R-x)(2R-x) > R^2;$$

$$\text{whence, } x < R.\frac{3-\sqrt{5}}{2}.$$

Thus it appears, that if a spherical dome, of uniform thickness, is to support itself, it cannot comprise the whole hemisphere; its sagitta cannot, indeed, be greater than $R.\frac{3-\sqrt{5}}{2}$, or $0.382R$; whence, the generating arc of the dome cannot be an arc of more than $51° 49' 50''$.

CHAP. X.

ON THE EQUILIBRIUM OF VAULTS, TAKING INTO ACCOUNT FRICTION.

641. Arches and domes, formed according to the rules prescribed in the preceding Chapters, owe their solidity to the stress, exerted against each other by the wedges which compose them, whilst, tending all of them to descend by their own weight, they mutually hinder that descent, and support one another. Thus, the structure will be kept firm, even although the wedges are not fastened together by any kind of cement, and even although their faces are so smooth and plane, that they might slide freely, without meeting with the smallest obstacle from friction.

But if we begin to consider the resistances, which nature, or art, can oppose to the slipping of the wedges, we shall find that we may give to arches and to domes, figures different from those which have been prescribed, and still sufficiently firm; because the want of equilibrium, amongst the weights of the wedges, may be compensated by the opposition of the resistances. We shall here undertake to examine the effect of friction; and we shall see to what degree it may be allowable, in virtue of this resistance, to alter the figures of equilibrium, which we have assigned to arches.

642. *Proposition I.* It being necessary, in order that the arch $AamM$, (Fig. 46.) may not move, sliding along the basis Mm, that (617.) $\Pi = A \,.\, \tan.\, e$, it will suffice, if friction be taken into the account, that Π shall lie between the determinate limits of the formula

$$\Pi = A \,.\, \frac{\tan.\, e \pm f}{1 \mp f \,.\, \tan.\, e}\,.$$

For the force, with which the arch tends to slide, descending from m towards M, is (589) $\Pi \cos.\, e - A \,.\, \sin.\, e$; and, therefore,

the force, with which it tends to slip, ascending from M towards m, is A sin. $e - \Pi$ cos. e. The friction opposes both these motions, with a force $= f . \Pi$. sin. $e + f . A$. cos. e. Therefore, for the state of equilibrium, we must have

$$\pm \Pi . \cos. e \mp A . \sin. e = f . \Pi . \sin. e + f . A . \cos. e.$$

$$\text{Wherefore } \Pi = A . \frac{\tan. e \pm f}{1 \mp f . \tan. e}.$$

643. *Coroll.* 1. Hence we see at once how it happens that the surface Kk may very well be horizontal.

We see also, that it is not necessary that the resultant GS shall fall, at right angles, upon the bed Mm, it being sufficient, (508) that the co-tangent of this angle shall lie between the limits $\pm f$.

644. *Coroll.* 2. We shall also have

$$d\Pi = \frac{A\,de}{\cos.^2 e} . \frac{1 + f^2}{(1 \mp f . \tan. e)^2},$$

and this value must be substituted in the equation (619) between z and e; and thus we shall have the limits between which the thickness of the arch may vary, the equilibrium being preserved.

645. *Coroll.* 3. Thus, in the example of Art. 622, for the circular arch, the equation will become

$$z = -R + \sqrt{\left\{R^2 + \frac{2Rm + m^2}{\cos.^2 e\,(1 \mp f . \tan. e)^2}\right\}}.$$

When R greatly exceeds the thickness m, we have, nearly,

$$z = \frac{m}{\cos.^2 e\,(1 \mp f . \tan. e)^2},$$

a formula, by which the thickness, proper for each point of the arch, may readily enough be computed. With the upper sign, we shall have the greatest thickness, and with the lower the least thickness, that can be given to the arch. It will be easy, from this formula, to make tables, and also to delineate, through points, the two curves or scales, the one of the greater, the other of the less thicknesses. And then, whatever be the external curve of the arch, provided that it does not go out of the limits of the two curves above-mentioned, the arch will be secure, as far as relates to this first condition of equilibrium.

646. *Coroll.* 4. In the scale of the lesser thicknesses there is a minimum value, where $\tan. e=f$, which gives $z=\frac{m}{1+f^2}$. And, in the scale of the greater thicknesses, there is a maximum, where $\cot. e=f$, which makes z infinite. Again, at the point where $e=90^{\circ}$, that is, at the horizontal support, both the curves give the same value of $z=\frac{m}{f^2}$.

Thus, if $f=0.75$, the greatest diminution of thickness, of which the arch can admit, will be at the distance of about 37° from the vertex, where $z=0.64m$; so that there it may be reduced to two-thirds of its thickness at the key-stone; and the greatest increase of thickness, of which it admits, will be at about 53° from the vertex; where its thickness may be augmented indefinitely.

647. *Proposition II.* In order that the arch $AamM$ (Fig. 46,) shall not move, turning about the points M, m, the condition, explained in Art. 623, must be fulfilled; except that, as the resultant GS may fall upon Mm with any obliquity, comprised within the limits (643) of an angle having $\pm f$ for its co-tangent, the straight lines Mi, mh, (Fig. 47.) must be inclined to Mm at an obtuse angle, which has $-f$ for its co-tangent. And it will be sufficient to take care, that the vertical, drawn through the centre of gravity of the arch, does not pass without the trapezium $fghi$.

648. SCHOLIUM. The equilibrium of the arch having been secured, it remains to consider the stress, which it exerts against the supports. It rests upon them as if it were all one piece. If, therefore, the weight of the whole arch Kak' is $2R$, and the supports decline from the vertical at an angle E, each of them (138) will sustain the normal stress $\frac{R}{\sin. E}$; whence, the horizontal stress $=R$. cot. E, and the vertical stress $=R$. It will be easy, by means of Chap. II, Book III, to proportion the dimensions of the foundations to these stresses.

CHAP. XI.

ON THE EQUILIBRIUM OF VAULTS, TAKING INTO ACCOUNT THE TENACITY OF CEMENTS.

649. After friction, the tenacity of cement offers itself to our consideration; which, holding together the wedges, and preventing them from sliding over each other, effectually concurs in maintaining the equilibrium and the stability of vaults. If this resistance were insuperable, the arch would be as it were all one mass; and, whatever were its form, it would be always sufficiently secure, provided that the piers had strength enough to sustain it.

But, although such a resistance, on account of the want of strength and durability in cements, cannot be considered at all as nearly inseparable, yet it has necessarily this effect, that when at length, the arch gives way, it will not break in all its sections, as an arch composed of several wedges would do, but will yield first in its weakest sections, the whole mass of the arch separating into three, or, at most, into four pieces. For the arch cannot, at first, give way, except either in two sections, as in Bb, Dd, (Fig. 50.) or else in three sections, as in Bb, Cc, Dd.

Hence, there is opened to us a way sufficiently easy, to examine, under this point of view, the firmness of any proposed arch, and also the force of the piers to support it: which is, by considering, separately, these two manners, in which the arch can separate; and, by comparing, for each of these dislocations, the forces that tend to cause it, with those that oppose it.

650. That disjunction of the arch, which opens only the two sections Bb, Dd, takes place, whenever the upper arch BCD descends in one piece, displacing, by its lateral stresses,

the planes Bb, Dd. That disjunction, again, which, together with the side sections, opens also the top Cc, takes place, whenever the point C descends, turning about the points B, D; whilst these points either give way laterally, or rise, turning about the extreme points A, E. In this case, the arch opens within at c, and outwards at b, d; exactly as if the polygon $ABCDE$ went to pieces, the angle C, at the top, opening, and the lateral angles, B, D, shutting themselves.

We shall proceed to investigate, for both these dislocations, the conditions of equilibrium. And, first, from the centres of gravity G, O, of the portions BC, AB, we shall suppose the verticals GR, OT, to be let fall; and from the points B, D, the straight lines BV, DV to be drawn, perpendicular to the sections Bd, Dd, which sections, produced, meet in N.

651. *Proposition I.* If the arch BCD (Fig. 50.) tend to fall vertically, removing the sections Bb, Dd, and if O be the weight of the solid AB, and G the weight of the solid BC, the equilibrium will be determined by these two equations,

$$f\,.\,O = G\,.\left(\frac{KN}{BK} - f\right),$$

$$O\,.\frac{AT}{BM} = G\left(\frac{KN}{BK} - \frac{AM}{BM}\right).$$

For, $2\,G$ being the weight of the arch BCD, this weight may be understood to be applied to the point V, whence it presses normally against the planes Bb, Dd, exerting itself in the directions VB, VD. The horizontal stress, which it exerts on B, will (596) be $= G\,.\,\tan.\,BVK = G\,.\,\cot.\,BNK = G\,.\,\frac{KN}{BK}$; and the vertical stress $= G$.

Now the first stress tends to push out the solid AB horizontally; which motion is resisted by friction, with a force $= f\,.\,G + f\,.\,O$: whence we have the first equation.

Further, the horizontal stress tending to overturn the solid AB, about the angle A, acts with the momentum $G.\frac{KN}{BK}.BM$,

and is resisted by the vertical stress G, with the momentum $G.AM$, and by the weight O, with the momentum $O.AT$. Equating the momentum of the stress to those of the resistance, and dividing by BM, we obtain the second equation.

652. *Proposition II.* If each of the two pieces BC, DC, (Fig. 50.) tend to turn about the vertex C of the arch, removing the points B, D, the equilibrium will be determined by the two following equations,

$$f.O = G.\left(\frac{BR}{CK} - f\right),$$

$$O.\frac{AT}{BM} = G\left(\frac{BR}{CK} - \frac{AM}{BM}\right).$$

Considering the arch (650) as a polygon $ABCDE$, moveable about its angles, we may (164) distribute the weights G, O, upon the angles C, B, A, E, and thus, following the notation of Art. 599, we shall have

$$2P = 2G.\frac{BR}{BK}; \quad Q = G.\frac{Rk}{BK} + O.\frac{AT}{AM};$$

$$V = O.\frac{TM}{AM}; \ \tan. m = \frac{Bk}{CK}; \ \tan. n = \frac{AM}{BM}.$$

Wherefore (599) the horizontal stress at the extreme point A will be $= G.\frac{BR}{CK}$, and the vertical stress $= G + O$; whence, the friction $= f.G + f.O$. Hence, in order that the portion AB may not move in an horizontal direction, $f.G + f.O$ must be equal to $G.\frac{BR}{CK}$; which is the first equation.

Further, it will (599) be requisite, for a state of equilibrium, that

$$P.\tan. m = (P + Q)\tan. n.$$

Whence, if the assigned values be substituted, we shall obtain the second equation.

653. *Coroll.* Hence, the method of examining the firmness of a given arch is evident. It will be proper, for various points

B of the arch, to calculate the equations laid down; in which, if the first members, that represent the action of the resistance, are found to be greater throughout than the second, which give the action of the stress, we may promise ourselves the complete security of the arch. The weakest section of all will correspond to that point B, for which the excess of the resistance above the stress, or that of the respective momentums, appears the least.

But if the other dimensions were given, and if it were required to find the thickness proper for the pier, the same equations would serve. The easiest method will be, first to assume an arbitrary thickness for the pier, and thence to seek the place B of the weakest section. Then, by the equations relating to that point B, we shall readily find the required thickness. The same may be said of any other dimension of the arch, or of the pier, which it may be wished to determine such as it ought to be for stability.

The same rules obtain, and the same equations, for domes; only that, instead of the profiles AB, BC, BCD, we shall have to consider the ungulas generated by the revolution of these same lines about the axis of the dome.

654. SCHOLIUM 1. It may happen in arches, but not in domes, that the two movements indicated in Art. 650, take place in opposite directions; the piers being overturned inwards, about the angles a, e. In the same manner as before, will the equations of equilibrium be found, relative to these motions. But, ordinarily, the danger of this happening is so small, that it is not worth the pains of employing ourselves upon it.

655. SCHOLIUM 2. The determination of equilibrium which is deduced from Prop. I, is that, which De la Hire had already proposed, and which has been explained, with greater clearness, by Couplet, by Belidor, and lastly, by Bossut (*Mem. de l'Acad. de Par.* 1774.). Coulomb, as far as I know, was the first (*Mem. Presentées*, &c. tom. VII.) to raise a suspicion, that this rule is insufficient, and that the vault may break in four pieces, instead of three; for which rupture Mascheroni (*Nuove Ricerche*, &c. Prob. X.) afterwards assigned the conditions of equilibrium, under the form, which we have expounded in Prop. II.

CHAP. XII.

ON THE FIRMNESS OF MODELS.

656. From an experiment made upon the firmness of the model of a machine, or of an edifice, we ought not to infer the firmness of the edifice, or of the machine itself, without great caution.

And, first, it is proper to distinguish between those stresses, which tend simply to displace the parts, and those which tend to break them. Transferring the system from a small to a large scale, the former stresses increase in an equal proportion with the resistances, the latter increase in a greater proportion; so that, with respect to these latter, in proportion as the structure is greater, it is weaker.

Again, amongst these same forces it is proper to distinguish three classes: some of them tend to draw asunder the parts, others to break them transversely, others to crush them by compression. To the first class belongs the stretching suffered by the key-stones, or bonds of vaults, and the planks of roofs; to the second, the load which tends to bend or break horizontal or inclined beams; to the third, the weight which presses vertically upon walls and columns.

657. *Proposition I.* If the side of a model be to the corresponding side of the structure as 1 to n, the stress, which tends to draw asunder, or to break transversely, the parts, increases, from the smaller to the greater scale as 1 to n^3; and the resistance to these ruptures increases only as 1 to n^2.

The first part of the proposition is manifest. For the stresses must increase as the weights of the parts which cause them, and these increase in the triplicate ratio of the homologous sides.

Again, the dimensions of the modular solid being a, b, c, those of the structure will be na, nb, nc. Now the absolute resistance, which opposes the drawing force, increases (533) in the ratio of ab to $na \times nb$; and the relative resistance, which opposes the force tending to break the parts transversely, increases (541) in the ratio of $\frac{a^2 b}{c}$ to $\frac{n^2 a^2 . nb}{nc}$: and both these ratios are reducible to that of 1 to n^2.

658. *Coroll.* 1. Wherefore, those parts which are subject to stresses of this kind, will have so much the less firmness in the structure, than in the model, as the number n is greater; which may increase, until the structure can no longer hold together.

659. *Coroll.* 2. In reasoning, then, from the model, from a small to a large scale, there is a certain limit of increase, which cannot be passed.

Let P be the greatest weight, which one of the small beams of the model can bear, aud p the weight, or stress, which it actually sustains. The greatest weight, which the corresponding beam of the structure can bear, will be $n^2 P$, and the weight, which it will actually sustain, will be $n^3 p$. Therefore, we must have $n^3 p < n^2 P$, and, at most, $n^3 p = n^2 P$; whence $n = \frac{P}{p}$. And this will give the greatest magnitude, of which the corresponding piece in the structure will admit.

660. *Coroll.* 3. But if we increase the model at pleasure, preserving in it a sufficient firmness, it will be necessary, in augmenting each piece, to alter some one of its dimensions.

Suppose, for example, that it is wished to preserve in a beam the breadth nb, and the length nc, proportionate to the model; and let it be required to find the necessary thickness xa, so that its resistance may be equal to the stress, which it has to sustain.

The resistance of the beam will be to that of its model as $\frac{x^2 a^2 . nb}{nc} : \frac{a^2 b}{c}$, or as $x^2 : 1$. Hence, the beam can, at the

most, sustain the weight Px^2, and will, in fact, sustain the weight n^3p. Wherefore, we must have, at the least,

$$Px^2 = n^3p, \quad \text{or} \quad x = n \,.\, \sqrt{\frac{np}{P}}\,.$$

If the section of the beam were a square, and if, retaining the length nc, we sought the side xa, it would be found, in the same manner, that $x = n \sqrt[3]{\frac{np}{P}}$.

661. *Proposition II.* The side of a model being to the corresponding side of the structure as 1 to n, the stress which tends to crush the parts, by compression increases, from the smaller to the greater scale, as 1 to n^3; and the resistance increases only as 1 to n.

For (566) the resistance to compression increases in the ratio of $\frac{a^2 b}{c^2} : \frac{n^2 a^2 \,.\, nb}{n^2 c^2}$, which is the ratio of $1 : n$.

662. *Coroll.* 1. Let P be the greatest load, which a modular wall, or column, can carry, and let p be the weight with which it is actually loaded. Then will the greatest load, which the corresponding wall, or column of the structure can carry, be nP; and n^3p will be the weight, which it will really sustain. Hence, we must have, at most, $n^3p = nP$; whence $n = \sqrt{\frac{P}{p}}$, for the greatest limit of increased dimensions.

663. *Coroll.* 2. But if, retaining the length, or height, nc, and the breadth nb, we wished to give to the solid such a thickness xa, as that it should not break in consequence of its increased dimensions, we should find, as above (660), $x = n^2 \sqrt{\frac{p}{P}}$, at the least.

And, in the case of a pilaster with a square base, or of a cylindrical column, if we sought, in like manner, the side, or the diameter, xa, we should find $x = n \sqrt[3]{\frac{n^2 p}{P}}$.

664. *Coroll.* 3. The principles here expounded will furnish us with a safe rule, to judge of the strength of a structure, from trials made upon its model. If, in the model, the strength of each piece has been proved, and if the limit of augmentation, of which it admits, has also been found, it is plain, that, when the whole model is increased, we must stop below the least of these limits. And, when it is wished to go beyond that limit, we have seen how the dimensions of each piece may be altered, so that the necessary firmness shall be preserved.

BOOK V.

ON MACHINES.

SECTION I.

ON MACHINES IN A STATE OF EQUILIBRIUM.

CHAP. I.

ON THE LEVER.

665. A LEVER is a rigid bar, moveable about a *fulcrum,* and subject to the action of various forces, at various points.

The distances of the directions of these forces from the fulcrum, are called the respective *Arms of the Lever.*

It is usual, in treating of the lever, to consider specially, the most simple case, in which a *weight* Q is in equilibrium with a *power* P. And the lever is said to be *of the first kind,* when the fulcrum lies between the power and the weight; *of the second kind,* when the weight is in the middle; and *of the third kind,* when the power is in the middle.

666. *Proposition.* If a lever is in a state of equilibrium, the power is to the weight in an inverse ratio of the respective arms of the lever.

Let a be the arm of the power P, and b the arm of the weight Q; then, in a state of equilibrium

$$P : Q :: b : a.$$

For (103) $P.a$, $Q.b$, are the momentums of the forces P, Q, to turn the lever about the fulcrum, and these, in the case of an equilibrium, must (105) be equal to one another. Wherefore, since $P.a = Q.b$,

$$P : Q :: b : a.$$

667. SCHOLIUM 1. This proposition, that two weights are in equilibrium, when their distances from the fulcrum are reciprocally proportional to the weights themselves, contains the principle and the foundation not only of the properties of the lever, but of those, also, of all the other machines; which may all of them, as we shall see, be reduced to the lever. Further, from this proposition, as from an universal principle, may the whole of Statics be deduced. Its truth was known by the most ancient writers on Mechanics, although the first, who demonstrated it, was Archimedes; whose proof, independent of every other mechanical principle, proceeds nearly in the following manner.

Let the lever XZ (Fig. 51.) be bisected by the fulcrum F, and loaded with weights, uniformly distributed throughout its whole length. It will, manifestly, be in equilibrium.

Let, now, any point L be taken in the lever, and let all the weights, uniformly distributed through the portion XL, be supposed to be collected in a single weight P, placed in A the bisection of XL. Similarly, let all the weights, uniformly distributed through LZ, be supposed to be collected in a single weight Q, placed in B the bisection of LZ. It is plain that the equilibrium will still subsist.

Wherefore, $P : Q :: XL : LZ.$

But $XL = XZ - LZ = 2\,FZ - 2\,BZ = 2\,BF$;

and $LZ = XZ - LX = 2\,FX - 2\,AX = 2\,AF.$

Wherefore, $P : Q :: BF : AF.$

668. SCHOLIUM 2. It has been objected to this proof, that to equate the force of several weights, uniformly spread over a line, with that of a single weight equal to their sum, and placed in the bisection of the line, is not au assumption so

evident, as to need no demonstration itself. We may, however, easily satisfy ourselves (Vince, *Phil. Trans.* 1794.) that the weight Q, placed in B, is equivalent to two weights q, q, each the half of Q, placed in the points L, Z, equidistant from B.

For if there be taken $Zf = LF$, and another fulcrum be supposed to be placed at f, it is evident, that, whether the weight Q be at B, or the weights q, q, at L and Z, in both cases will this fulcrum support the half of the weight Q. Wherefore, if the fulcrum be removed, the same force will be required at f, to balance the weight Q, as to balance the two weights q, q. Wherefore, &c.

669. *Coroll.* 1. If the forces are parallel, in the first kind of lever, the power may be greater or less than the weight; but, in the second kind, the power is always less than the weight, and in the third kind, it is always greater.

670. *Coroll.* 2. If the weight V of the lever itself is to be taken into account, this will be a third force acting on the lever, and its arm will be the distance of the fulcrum from the centre of gravity of the lever.

Let c be this arm. Then will the equation of equilibrium be

$$Pa \pm Vc = Qb;$$

accordingly as the weight V tends to turn the lever in the direction of the power P, or in a contrary direction.

671. *Coroll.* 3. A lever of the second kind being suposed to have weight, and to be lengthened, whilst, on the one hand, the force of the power is increased, by the increase of the arm a, on the other hand its effect is diminished, on account of the increase of the momentum Vc. Hence, this problem.

The weight Q and its arm b being given, of a lever of the second kind, and also the weight g of a portion of the lever, of the length of the standard unity, to find the whole length a of the lever, so that the power may act to the greatest advantage.

We shall have $V = ga$, and $c = \frac{1}{2}a$. Hence, for the equilibrium,

$$Pa - \tfrac{1}{2}ga^2 = Qb, \text{ and } P = \tfrac{1}{2}ga + \frac{Qb}{a}.$$

When P is a minimum, $dP = 0$;* whence, $a = \sqrt{\frac{2Qb}{g}}$.

672. *Coroll.* 4. Thus, the lever having weight, the fulcrum sustains the force of the powers P, Q, V, as if they were immediately (128) applied to it.

673. SCHOLIUM. Whatever be the number and the directions of the forces applied, and whatever be the manner, in which the lever is connected with the fulcrum, the properties of equilibrium, and the pressures of the point of support, must be deduced from the conditions of equilibrium of a system of invariable form. The reader is referred, therefore, to Chap. XVI. and XVII. Book I.

We shall, likewise, refer to the last Chapters of Book III. for all that relates to the ascertaining whether the fulcrum, and the lever itself, be sufficiently strong to sustain the stresses of the powers: and the same remark is applicable to the other machines, also, which we shall proceed successively to consider.

CHAP. II.

ON THE BALANCE.

674. *Proposition I.* In the balance, where any two equal weights whatever, placed one at each extremity, are to be in equilibrium, it is requisite, first, that the balance preserve its equilibrium when it carries no weights; secondly, that the two arms be of equal length.

For, let M, N, be the momentums of the two parts of the balance, when it has no weights appended; a, b, the lengths of the two arms; P, P, the weights which are in equilibrium upon it. Then, we must have

$$M + Pa = N + Pb;$$

$$\text{whence, } P(b-a) = M - N.$$

And, since this equation must hold, whatever be the value of P, we have, necessarily, $M = N$, $a = b$.

675. Scholium 1. If either of the two specified conditions is wanting, the balance is said to be *false;* but it may still be made to serve justly to determine the weight.

For the first defect is easily corrected, by giving an equipoise to the naked balance, through the addition, in one of the scales, of such a weight as will produce an equilibrium.

The second defect is obviated, by placing the body to be weighed, first in the one scale, and then in the other: its just weight will be a mean proportional between the two different marked weights, that have served to balance it. For, if the weight P, placed successively in the two scales, is balanced by the two marked weights Q, Q', we shall have $P.a = Q.b$, $P.b = Q'.a$; whence, $P^2 = Q.Q'$, and $Q : P :: P : Q'$.

676. Scholium 2. There is, also, another mode of making a false balance serve exactly to ascertain the equality of two weights. This consists in placing the two weights successively in the same scale of the balance, and counterpoising them by the same weight placed in the other scale. If the two weights

have, in the two successive weighings, been in equilibrium with the same counterpoise, they are necessarily equal to one another.

Let the first weight be P, and let it be in equilibrium with the marked weight Q. We cannot conclude, at once, that $P = Q$, because we are supposed not to be sure of the justness of the balance. Let the second weight be P', and let this, also, be in equilibrium with the same marked weight Q; and here, too, for the same reason, we cannot say that $P' = Q$. But we may, with certainty, affirm $P = P'$; because, these two weights, being in equilibrium with the same marked weight Q, under the same circumstances, cannot but be equal.

Since it is physically impossible to secure, with the utmost degree of precision, the perfect equality of the lengths a, b, of the arms, and of the momentums M, N, it will always be well to employ this method of the two weighings, when it is wished scrupulously to ascertain the equality of two weights.

677. Scholium 3. The perfection of the balance, and the facility of managing it, require other conditions. In the first place, for every the smallest inequality of the two weights, the needle of the balance should slowly incline itself, stopping in a position oblique to the horizon, without going completely down. Further, if the balance is in a state of equilibrium, and if the needle is accidentally made to incline itself, it ought not to stop in that inclined position, nor completely to go down, but slowly to reassume its vertical position.

The fulfilment of these conditions depends, above all, upon the relative position of the three points O, C, G, (Fig. 52) of which O is the centre of motion, G the centre of gravity of the beam of the balance, C the intersection of the vertical line OG with the horizontal line AB, which joins the points of suspension of the two scales. The two following propositions will enable us to determine the most advantageous situation of these points.

678. *Proposition II.* The equilibrium of a balance having been disturbed, by the addition of a very small weight p to one of the equal weights P, P, that are weighed in it, to determine the inclination, which will thence ensue, in the needle of the balance.

Suppose that, by the addition of the small weight p to the weight P, hanging from the point B, the beam passes into the position aOb; and let the angle $AOa = BOb = GOg = \psi$, and the angle $AOC = BOC = \alpha$: from the points a, b, g let there be drawn ah, bk, gi, perpendicular to OG, and let M be the weight of the beam.

Then, for the balance to be in equilibrium in the position aOb, we must have

$$P \cdot ah + M \cdot gi = (P + p)\, bk.$$

Now,
$$\begin{aligned} ah &= AO \sin.(\alpha + \psi) = AO \sin.\alpha \cos.\psi + AO \sin.\psi \cos.\alpha \\ &= AC \cos.\psi + OC \sin.\psi, \\ bk &= AO \sin.(\alpha - \psi) = AO \sin.\alpha \cos.\psi - AO \sin.\psi \cos.\alpha \\ &= AC \cos.\psi - OC \sin.\psi, \\ gi &= OG \sin.\psi. \end{aligned}$$

Substituting these values in the equation of equilibrium, we shall have

$$\tan.\psi = \frac{p \cdot AC}{(2P + p)\, OC + M \cdot OG};$$

or, putting M' for the weight of the loaded balance, so that $M' = M + 2P + p$, we shall have more briefly,

$$\tan.\psi = \frac{p \cdot AC}{M' \cdot OC + M \cdot CG}.$$

679. *Coroll.* 1. The mobility of the balance, therefore, will be the greater, the longer are its arms, the nearer together are the three points O, C, G, and the less it is loaded.

680. *Coroll.* 2. If $OC = O$, $\tan.\psi = \dfrac{p \cdot AC}{M \cdot CG}$; and then the mobility of the balance is the same, whatever be the load which it carries.

681. *Coroll* 3. If, at the same time, $OC = O$, and $GC = O$, so that the three points O, C, G, coincide, $\tan.\psi$ will be infinite; and then for every the smallest derangement of the equilibrium, the needle of the balance will go down, and describe a quadrant of a circle, so as itself to become horizontal. And if OC and CG were negative, $\tan.\psi$ would become negative,

and the needle, still going down, would describe more than the quadrant of a circle. It is manifest that these constructions of the balance are faulty.

682. *Proposition III.* If the needle of a balance, which is in perfect equilibrium, be made to incline at the angle ψ, to find the force, with which the beam tends to resume the position of equilibrium.

Let S be the momentum of inertia of the loaded balance, with respect to the axis of motion. The accelerating force, with which the beam will tend to turn round in the direction $aAbB$, to place itself again in the position AOB, will be (341)

$$= \frac{P \cdot ah - P \cdot bk + M \cdot gi}{S}:$$

And, substituting the values (678) of ah, bk, gi, the force sought will be found to be

$$= \frac{\sin. \psi}{S}(2P \cdot OC + M \cdot OG)$$

$$= \frac{\sin. \psi}{S}(M' \cdot OC + M \cdot CG).$$

It must be observed, in computing the momentum of inertia S, that the masses of the two weights P, P, which hang freely from the points A, B, must be understood to be concentered in those points.

683. *Coroll.* If $OC = CG = O$, the restoring force vanishes; in that case, the balance is said to be *indolent;* indeed it stops altogether, the needle resting indifferently in any position. But if OC and CG are negative, the restoring force is negative; and then the balance is said to be *unstable,* because, for every the smallest inclination of the needle, instead of resuming the position of equilibrium, it deviates more and more from it, and goes completely down.

CHAP. III.

ON OTHER FORMS AND COMBINATIONS OF THE LEVER.

684. ***Proposition I.*** In the steelyard, if the longer arm be divided into equal parts, and if the moveable standard weight pass successively through each division, it will be in equilibrium with weights increasing in arithmetic progression.

Let a be the shorter arm; b, b', b'', &c. the successive arms of the lever, at the ends of which the moveable weight Q acts; P, P', P'', &c. the weights with which it is in equilibrium; $V.c$ the momentum of the weight of the steelyard itself. We shall have, successively,

$$\begin{aligned} Pa &= Qb + Vc, \\ P'a &= Qb' + Vc, \\ P''a &= Qb'' + Vc, \\ \&c. &= \&c. \end{aligned}$$

where, if b, b', b'', &c. are in arithmetic progression, it is plain that P, P', P'', &c. must likewise be so.

Hence, the graduation of the steelyard is a matter of the greatest facility.

685. *Proposition II.* If a power be in equilibrium with a weight, by means of several levers, the one acting upon the other, the power is to the weight as the product of all the arms of levers lying towards the side of the weight, is to the product of all the arms of levers lying towards the side of the power.

First, let there be three levers; let X be the force, which the first lever exerts on the extremity of the second; Y the force, which the second exerts on the extremity of the third; let a, b, be the arms of the first lever; a', b', the corresponding arms of the second; a'', b'', those of the third. Then we shall have

$$Pa = Xb;\ Xa' = Yb';\ Ya'' = Qb'';$$
$$\text{whence, } Pa.a'.a'' = Q.b.b'.b''.$$

And, in the same manner may the proposition be proved for any number of levers, so acting upon one another.

686. *Proposition III.* To determine the conditions of equilibrium in the drawbridge.

In the drawbridge, the profile of which is delineated in Fig. 53, two levers are combined, the one of the first kind, the other of the second kind, AB is the floor of the bridge, moveable about the fulcrum A, having its extremity B connected, by the chain BC, with the extremity C of the lever KDC, moveable about the fulcrum D.

If the two levers KDC, AB, acted immediately upon each other, the condition of equilibrium would readily be obtained, from the preceding proposition. But it is necessary to consider the weight, and the position of the chain BC. We shall put X for the tension of the chain, and expressing, by the letters marked in the figure, the weight of the sides AB, BC, DC, DK, we shall distribute each of these weights upon the points of support, and upon the extremities of the sides. Thus, it is manifest that the point B will remain loaded with the weight $T+R$; the point C with the weight $Q+R$; the point K with the weight P.

Through the points A, D, let there be drawn AG, DF, perpendicular to BC, and AH, LO parallel to the horizon; and, through the points B, C, K, let there be drawn, to AH and LO, the vertical lines BH, CO, KL.

Now, it is plain, that in the lever AB, the weight $T+R$, at the end of the arm AH, is in equilibrium with the tension X, acting at the end of the arm AG; and, in the lever KDC, the weight P, at the end of the arm DL, is in equilibrium with the weight $Q+R$, at the end of the arm DO, together with the tension X, at the end of the arm DF. Wherefore,

$$(T+R).AH = X.AG;$$

$$\text{and, } P.DL = (Q+R).DO + X.DF;$$

from which two equations if X be eliminated, we shall have the required condition of equilibrium.

687. *Coroll.* If the quadrilateral figure $ABCD$ is a parallelogram, we have

$$P . DK = (T + Q + 2R) . AB.$$

And the weight P, determined by this equation, will balance the drawbridge in any position whatever.

688. SCHOLIUM. The chain BC, curving itself by its own weight, will not be stretched into a straight line, as we have supposed it to be: but this will not cause much variation.

We have also supposed, that the weights, which load the sides, are so distributed, that they may be considered as collected in the bisections of the sides. But it is easily seen, how the conditions of equilibrium must be changed, if the weights are differently distributed.

CHAP. VI.

ON THE WHEEL AND AXLE.

689. In the case of the wheel and axle (Fig. 54.) the power P is applied to the periphery of a wheel, and the weight Q hangs by a rope, wound about a cylinder, which turns round together with the wheel.

Sometimes, the cylinder does not immediately support the weight; but (Fig. 55.) puts in motion a wheel with teeth, the cylinder or pinion of which acts, in a similar manner, upon another wheel; and so on to the last wheel, the cylinder of which carries the weight Q. These are what are called *Toothed-Wheels*, which, as is evident, are systems of several wheels and axles, acting upon each other.

690. *Proposition I.* When there is an equilibrium on the wheel and axle, the power is to the weight as the radius of the axle is to the radius of the wheel.

Let the radius of the wheel be a, that of the axle b; in the case of an equilibrium, we shall have

$$P \cdot a = Q \cdot b,$$

$$\text{whence,} \quad P : Q :: b : a.$$

This follows from Art. 105; and it is, besides, very evident that this machine is reducible to a lever of the first kind.

691. *Proposition II.* If a power be in equilibrium with a weight, by means of a system of toothed-wheels (Fig. 55.) the power will be to the weight as the product of the semi-diameters of the pinions is to the product of the semi-diameters of the wheels.

The demonstration is similar to that of Art. 685.

692. Scholium. If the weight Q descend, giving a rotatory motion to the system, it is often useful to know how many

turns the last wheel C makes, in the time, in which the first completes one revolution. Now, to know this depends upon our knowing the number of the teeth of each wheel A, B, and the number of the teeth of each pinion b, c. Neither the last wheel, nor the first pinion, are taken into the account; because these are not toothed, or at least it is of no consequence, if they are so.

Let A, B, be the numbers of the teeth of the wheels A, B, and b, c, the numbers of the teeth of the pinions b, c. Whilst the wheel A makes N turns, let the pinion b, with its wheel B, make N' turns, and let the pinion c, with its wheel C, make N'' turns. It is manifest, that, in proportion as a wheel has more teeth than the contiguous pinion has, so much the fewer turns will the former make, in the same time as the latter. Wherefore,

$$N : N' :: b : A,$$
$$\text{and } N' : N'' :: c : B;$$
$$\therefore\ N : N'' :: bc : A \cdot B.$$

693. *Coroll.* Hence the solution of this problem: "In a given system of toothed-wheels, to assign the number of the teeth of the wheels, and of the teeth of the pinions, so that the number of the contemporaneous revolutions, made by the extreme wheels, shall be to one another in a given ratio."

CHAP. V.

ON THE PULLEY AND THE TAGLIA.

694. In the *Fixt Pulley*, the centre is fixt to an immoveable support, and the power and the weight are applied to the two extremities of the string which goes round the pulley.

In the *Moveable Pulley*, the weight hangs from the centre of the pulley; one end of the string is fixt; and the other end is drawn by the power.

The *Taglia* consists of a system of fixt pullies, collected in one common block, and also of a system of moveable pullies, in a separate block, to which the weight is attached; with one string going round all the pullies, and having one of its ends fixt to a point of the system, and the other end, going from one of the fixt pullies, drawn by the power.

Sometimes several moveable pullies, or several taglias are combined, so that the one acts upon the other; and these are called *Compound Pullies*, or *Compound Taglias.*

It is manifest, that the fixt pulley is reducible to a lever, of the first kind, and with equal arms; so that it gives no advantage to the power, and only serves to change the power's direction. But the moveable pulley, as we shall soon see, is reducible to a lever of the second kind.

695. *Proposition I.* When the moveable pulley (Fig. 56.) is in a state of equilibrium, the two directions of the string decline equally from the vertical; and the power P is to the weight Q, as radius to twice the cosine of this declination, or, as the radius of the pulley to the chord of the arc in contact with the string.

For the weight Q being opposed by the tensions of the two parts PA, EB of the string, the two tangents PA, EB, produced, must necessarily meet in a point X of the vertical drawn

through the centre of the pulley. Hence, the angles AXC, BXC, must be equal, and (148),

$$P : Q :: 1 : 2 \cos. AXC :: 1 : 2 \cos. CAD :: AC : AB.$$

The same conclusion is arrived at, by considering the moveable pulley as a lever of the second kind, in which the power P tends to lift up the pulley, making it turn about the point B, whilst the weight Q tends to turn it, in a contrary direction. Drawing BF perpendicular to PA, the momentum of the power, $P . BF$, ought to be equal to the momentum of the weight, $Q . BD$. Wherefore,

$$P : Q :: BD : BF;$$

or, by the similar triangles BAF, ADC,

$$P : Q :: AC : AB.$$

696. *Coroll.* The moveable pulley adds to the power, so long as the angle AXC is less than 60°; beyond which, it is disadvantageous. If the two opposite parts of the string are parallel, the power is the half of the weight: and this is the greatest advantage that can be gained in the pulley.

697. *Proposition II.* When the taglia is in a state of equilibrium, (Fig. 57.) the branches of the string, which sustain the moveable pullies, decline from the vertical, so that the sum of the sines of these declinations is equal to nothing; and the power is then to the weight, as radius is to the sum of the cosines of these declinations.

For the tensions of the branches of the string will, (176.) be all equal to one another; and resolving each of them into two forces, the one horizontal, the other vertical, the horizontal forces must counterbalance each other, and the vertical forces must be in equilibrium with the weight Q. Now, taking for a common radius the straight line which expresses the tension of the string, the former forces will be expressed by the sines of the respective declinations, and the latter by their cosines. Wherefore, the sum of the sines must be equal to nothing, and the sum of the cosines must be equal to the weight Q. Therefore the tension of the string, or the power P, will be to the weight Q, as radius is to the sum of the cosines.

698. *Coroll.* If the strings are parallel, the power is to the weight as unity to the number of strings, which draw the moveable block; and this is the most advantageous disposition of such a system.

699. *Proposition III.* In a system of moveable pullies, in which the one acts upon the other, (Fig. 58.) the power is to the weight, as the product of the semi-diameters of the pullies, is to the product of the chords of the arcs in contact with the string, in each pulley.

The proof is similar to that of Art. 685.

700. *Coroll.* If all the strings are parallel, which is the most advantageous disposition, the power is to the weight as $1 : 2^n$; n being the number of the pullies.

701. Scholium. Several taglias may also be combined, so that they shall act upon one another, the power being applied to the first of them, and the weight to the last. Here, the ratio of the power to the weight will be compounded of the ratios which obtain for each of the taglias.

CHAP. VI.

ON THE INCLINED PLANE.

702. *Proposition.* If a power P sustains a weight Q, placed upon an inclined plane, the power is to the weight, as the cosine of the plane's inclination to the vertical, is to the cosine of the inclination of the power to the plane itself.

Let the angle, which the plane makes with the vertical $= m$, and the angle, made by the direction of the power with the plane $= n$; then will the equation of equilibrium be

$$P = Q.\frac{\cos. m}{\cos. n}.$$

For, resolving each of the forces P, Q, into two, the one parallel to the plane, the other perpendicular to it, the forces parallel to the plane will be $P.\cos. n$, $Q.\cos. m$; and the normal forces will be $P.\sin. n$, $Q.\sin. m$. Now, these latter are destroyed by the plane itself, which is pressed with a force

$$= P.\sin. n + Q.\sin. m;$$

and the former two forces ought to destroy each other. Wherefore

$$P.\cos. n = Q.\cos. m,$$

$$\text{and } P = Q\frac{\cos. m}{\cos. n}.$$

703. *Coroll.* When the power acts horizontally, the angle n becomes the complement of the angle m; whence,

$$P = Q.\cot. m;$$

and, therefore, the power is to the weight as the height of the plane is to its base.

If the direction of the power is parallel to the plane, the angle $n = 0$; whence,

$$P = Q.\cos. m;$$

and, therefore, the power is to the weight, as the height of the

plane to the length; and this is the greatest advantage, which can be gained from the inclined plane.

704. SCHOLIUM. We have said, (667.) that the principle of the lever accounts for the equilibrium in all the other machines, which may all of them, therefore, be reduced to the lever. This is manifest, in the wheel and axle, and in the pulley; in the inclined plane it is not so evident. We shall proceed, therefore, to demonstrate the proposition of Art. 702, by means of the principle of the lever.

Let AFB, (Fig. 59.) be a circle, having its diameter AB horizontal, and let the semi-diameter CF be inclined to AB at an angle $FCB = m$. Through F let there be drawn the tangent RT, and the secant PS inclined to RT at an angle $PFR = n$. From the point F let there be drawn FN perpendicular to CB, and from C, CN perpendicular to PS. Thus,

$$\text{the angle } FCM = MFT = m;$$

$$\text{and the angle } FCN = PFR = n.$$

Let, now, CF be supposed to be a lever, moveable about C; and, at the extremity F, let there be applied a power P, which acts in the direction of the straight line FP, and a weight Q, which will act in the direction of the vertical FM. By the principle of the lever, there will be an equilibrium, when

$$P : Q :: CM : CN :: \cos. m : \cos. n.$$

Now, the weight Q, placed at F, tends to descend through the arc of the circle, or through the tangent, in the same manner as if it were placed on the plane RT, declining from the vertical at an angle $MFT = m$. And the power P acts in the direction FP, which makes with the inclined plane the angle $PFR = n$. Wherefore, when a weight Q, placed on a plane inclined to the vertical at an angle m, is kept in equilibrium by a power P inclined to the plane itself at an angle n,

$$P : Q :: \cos. m : \cos. n; \text{ wherefore, \&c.}$$

CHAP. VII.

ON THE SCREW.

705. An *Helix* is a curve described on the surface of a right cylinder, having a constant inclination to a straight line in the surface of the cylinder, drawn parallel to the axis, and which is termed a *side* of the cylinder. The distance between two consecutive points of the helix, taken upon any the same *side* of the cylinder is called the *step* of the helix.

Let the surface of the cylinder AC (Fig. 60.) be unrolled into the rectangle ac, in which are marked the parallels ah', hk', &c. When this rectangle is wrapped about the surface of the cylinder, the helix will be traced there. Whence it is manifest, that every element of the helix is a straight line, with such an inclination, that the height is to the base, as the step of the helix, is to the periphery of the base of the cylinder.

706. A *Screw*, (Fig. 61.) is a right cylinder furnished on its surface with a projection, which follows the course of an helix. It turns in a fixt cylinder M, called the *Female Screw*, having a hollow spiral corresponding to the projection of the screw. The power P, by means of a handle AB, keeps in equilibrium the screw, loaded at its top, with the weight Q, which tends to descend, taking a rotatory motion, through the threads of the screw.

Sometimes, the screw is fixt, whilst the female screw turns about it; to which, in that case, both the forces P and Q are applied.

A compound machine is formed of the screw, and of the wheel and axle, where the screw does not immediately sustain the weight, but urges the tooth of a wheel, (Fig. 62.) from the axle of which hangs the weight Q; and this is called the *Perpetual Screw.*

707. *Proposition I.* When there is an equilibrium in the screw, the power is to the weight, as the step of the helix is to the periphery, which the power tends to describe.

Let the step of the helix $= h$; let (Fig. 61.) $cA = a$, $cB = b$. For the power P, which acts at A, substitute a power X, which acts at the point B. Then, (107.)

$$P : X :: cB : cA,$$

$$\text{or, } P : X :: 2\pi b : 2\pi a.$$

Now X is an horizontal power, which sustains the weight Q, whilst it tends to descend through an inclined plane, where (705) the height is to the base, as h to $2\pi b$. Wherefore, (703)

$$X : Q :: h : 2\pi b;$$

and, compounding the two proportions, we have

$$P : Q :: h : 2\pi a;$$

whence, $P = \dfrac{Qh}{2\pi a}$, is the equation of equilibrium.

Hence, it is manifest, that the screw is a machine compounded of a lever of the second kind, and of an inclined plane.

708. Scholium 1. The same condition of equilibrium obtains, even when the axis of the screw is not vertical, provided that the resistance Q acts in the direction of the screw's axis, and the power P acts in the direction of a tangent to the circle which is perpendicular to the axis. If this be not the case, it will become necessary to resolve these two forces, and to consider only those of the compounding forces, which act in the above-mentioned directions; the other being destroyed by the points of support.

709. Scholium 2. If the power P increases beyond the value required for equilibrium, so that it raises the head of the screw, whilst it makes an entire revolution, the head of the screw will be raised through a space equal to the step of the helix. Hence, the use of a screw, with very strait threads, to measure very small motions.

710. *Proposition II.* When there is an equilibrium in the perpetual screw, the power is to the weight, as the product of the radius of the cylinder multiplied by the step of the helix is to the product of the radius of the wheel multiplied by the periphery which the power tends to describe.

The proposition is proved in a manner similar to that of Art. 685.

CHAP. VIII.

ON THE WEDGE.

711. *Proposition.* If the power P, (Fig. 63.) acts perpendicularly upon the head of the wedge ACB, the power is to the pressure which it exerts perpendicularly on each side of the wedge, as the head of the wedge is to its side.

Let A and B be the pressures exerted perpendicularly on the sides AC, BC; and from the vertex C let there be drawn CZ perpendicular to AB. Then, (139)

$$P : A : B :: \sin. ACB : \cos. BCZ : \cos. ACZ;$$

whence, $$P : A : B :: \sin. ACB : \sin. B : \sin. A,$$

or, $$P : A : B :: AB : AC : BC.$$

Hence it is evident, that, the altitude CZ being given, the sharper the wedge is, the greater will be its force.

SECTION II.

ON MACHINES IN A STATE BORDERING UPON THAT OF MOTION.

CHAP. IX.

ON THE EQUATION OF THE STATE BORDERING UPON MOTION.

712. If there were no resistance, the state of equilibrium hitherto considered, would still be the state nearest to that of motion. Every the least diminution of the power P would suffer the weight Q to fall, and every the least increase of P would raise it. But let us now suppose, that between the two forces P, Q, there is interposed the resistance, or passive force N, which acts with the momentum $N.k$, whilst the powers P and Q act with the momentums $P.a$, $Q.b$, respectively. Then, since the weight P is upon the point of raising up Q, we must have

$$P.a = Q.b + N.K;$$

and in order that the weight Q may be upon the point of descending, overcoming the power P, we must have

$$P.a = Q.b - N.k.$$

Wherefore, the general equation of the state nearest to motion, will be

$$P.a = Q.b \pm N.k.$$

The upper sign will give the greatest value of P, and the lower sign the least; and in all the intermediate values the equilibrium will subsist.

This equation obtains in the case of rotatory motions; when only progressive motions are considered, instead of the momentums $P.a$, $Q.b$, $N.k$, the forces themselves P, Q, N must be taken.

Thus much being understood, the equation of the state nearest to motion will easily be found for each machine; the difficulty being reduced to that of exactly determining the resistance N, or its momentum; of which we shall give an example, which may serve as an introduction to what is to follow.

713. *Proposition.* Let the power P, (Fig. 64.) sustain the weight Q by means of the rope POQ, applied to the arc AB of any kind of curve EOR; and let it be required to find the equation of the state nearest to motion; regard being had to the friction of the rope on the arc AB.

Let $BA=a$, $BM=s$, $Mm=ds$; and, the two normals MK', mK meeting in K, let the radius of curvature $Km=r$. Further, let the tension of the rope at the point $M=T$, the tension at $m=T+dT$, and let the pressure on the small arc $Mm=Nds$. Taking away the element of the curve Mm, and substituting for it a force $OZ=Nds$, it is manifest that an equilibrium will still subsist between the two nearly equal forces T, $T+dT$, and the force Nds, which makes equal angles with them. Wherefore, (148) $T=\dfrac{Nds}{2\cos. KOM}$; and considering KOM as a triangle, right-angled at M, we shall have

$$\cos. KOM=\frac{MO}{KO}=\frac{ds}{2r}.$$

Therefore, $T=Nr$.

Again, putting f for the coefficient of friction, $dT=\pm fNds$; and, dividing this by the preceding equation, we shall have

$$\frac{dT}{T}=\pm\frac{fds}{r}.$$

In integrating, the constant quantity must be determined, so that when $s=0$, $T=Q$; then, the integral will be completed by making $s=a$, and $T=P$.

714. *Coroll.* 1. Let EOR be an arc of a circle, so that r is constant. The equation of the state nearest to motion will become

$$P=Qe^{\pm\frac{fa}{r}}$$

715. *Coroll.* 2. If the rope be supposed to be wound about a cylinder, and to touch it in the half of its circumference, so that $\frac{a}{r} = \pi$, and if, we suppose that $f = 0.35$, we shall have, very nearly, $e^{\frac{fa}{r}} = 3$. Hence, in the state nearest to motion, the greatest value of P will be

$$P = 3\,Q, \text{ and the least value } P = \tfrac{1}{3}\,Q.$$

But if the rope make one turn and a half about the cylinder, these values will become

$$P = 27\,Q; \; P = \tfrac{1}{27}\,Q.$$

And if it make two turns and a half, then

$$P = 243\,Q; \; P = \tfrac{1}{243}\,Q.$$

And thus, at every additional turn, the value of P must increase nine times as much, to be upon the point of raising the weight, and it may be diminished in the same ratio, so as simply to sustain it.

Hence it appears, how enormous a force is required to raise a weight by means of a rope, which is wound many times about an immoveable cylinder; and how small a force will be sufficient to hinder the descent of the weight.

CHAP. X.

ON THE EQUATION OF THE STATE BORDERING UPON MOTION, IN THE LEVER AND IN THE WHEEL AND AXLE.

716. *Proposition I.* In the lever moveable about an axis, and acted upon by the parallel forces P, Q, applied at the extremities of the arms a, b, if r be the radius of the axis, the equation of the state nearest to motion will be

$$Pa = Qb + \frac{fr(P+Q)}{\sqrt{(1+f^2)}}.$$

Let the power P (Fig. 65.) be upon the point of raising the weight Q. Let the resultant ER of the two forces P and Q, which is equal to their sum, be resolved into the tangential force ES, and into the normal force ET. Then, if the angle $RES = m$, we shall have $ES = (P+Q)$ cos. m, and $ET = (P+Q)$ sin. m. Now, the two forces ES, ET, being substituted for the two P, Q, in the state nearest to motion, it will be necessary that the former of these be precisely equal to the friction; whence, $ES = f \,.\, ET$; or, cos. $m = f$. sin. m, and, therefore, sin. $m = \frac{1}{\sqrt{(1+f^2)}}$, and $ES = f \,.\, ET = \frac{f(P+Q)}{\sqrt{(1+f^2)}}$, which will be the value of the friction.

Thus, the two forces P, Q, and the resistance $\frac{f(P+Q)}{\sqrt{(1+f^2)}}$, acting at the distances a, b, r, it is manifest (712) that, in the state nearest to motion,

$$Pa = Qb + \frac{fr(P+Q)}{\sqrt{(1+f^2)}}.$$

717. *Coroll.* 1. The coefficient f is commonly so small, that f^2 may be neglected; whence the equation becomes

$$Pa = Qb + (P+Q)fr,$$

$$\text{or, } P = Q \cdot \frac{b+fr}{a-fr}.$$

718. *Coroll.* 2. If the directions of the powers P, Q, are not parallel, but meet at an angle θ, instead of the binomial $P+Q$, we must substitute in the preceding equations the value of the resultant, which (24) will be

$$= \sqrt{(P^2 + 2P \cdot Q \cdot \cos.\theta + Q^2)}.$$

719. *Coroll.* 3. If several levers, as three for example, act upon each other, using the same notation and the same mode of proceeding, as in Art. 685, we shall have the equations,

$$Pa = Xb + (P+X)fr,$$
$$Xa' = Yb' + (X+Y)fr',$$
$$Ya'' = Qb'' + (Y+Q)fr'';$$

whence, eliminating X and Y,

$$P = Q \cdot \frac{b+fr}{a-fr} \cdot \frac{b'+fr'}{a'-fr'} \cdot \frac{b''+fr''}{a''-fr''}.$$

And if $a = a' = a''$; $b = b' = b''$; $r = r' = r''$; we shall have

$$P = Q \left(\frac{b+fr}{a-fr}\right)^n,$$

n being the number of the levers.

720. *Proposition II.* In the wheel and axle, if the forces P, Q are parallel, if a be the radius of the wheel, b that of the cylinder, r that of the axis of rotation, the equation of the state nearest to motion, taking into account the friction and the rigidity of the rope, will be

$$Pa = Qb + (P+Q)fr + b(\mu + \nu Q).$$

Here the last term is owing to the rigidity of the rope, the which resistance is expressed (524) by $\mu + \nu Q$, and acts at the distance b. In other respects the equation is deduced as above (717).

If several wheels and axles are combined, calling X the force which the first exerts upon the second, Y that which the second

exerts upon the third, and so on, for each of these we may deduce its own equation, by proceeding as before.

721. SCHOLIUM 1. When the power P is not applied to a single point only of the circumference of the wheel, but is distributed amongst many points diametrically opposite to each other, as is commonly the case in the crane, then this power does not at all affect the friction, and the equation becomes

$$Pa = Q(b + fr) + b(\mu + \nu Q).$$

722. SCHOLIUM 2. All these equations serve for that state in which the power is upon the point of raising the weight; but if it is wished to adapt them to the state, in which the weight is ready to overcome the power, it will be sufficient to change the sign of the coefficients of the resistances f, μ, ν; which must be understood, also, in the following applications.

CHAP. XI.

ON THE EQUATION OF THE STATE BORDERING UPON MOTION IN TOOTHED WHEELS.

723. WHEN a power is to raise a weight, by means of a system of toothed-wheels, the friction, which the tooth of each wheel causes by rubbing against the tooth of the pinion, or against the spindle of the box, which it turns round, is not to be neglected.

724. *Proposition I.* If the power P, (Fig. 66.) at the arm of a lever AK, moves a wheel R, the teeth of which urging the spindle of the box, or the pinion D, turns round the box itself, and, at the same time, the cylinder B, from the circumference of which hangs the weight Q; in the state nearest to motion, it is required to find the value of the power P, taking into account the friction of the tooth against the spindle, and neglecting, for the present, all the other resistances.

Let D be the point in which the tooth touches the spindle; let the straight lines AD, BD be drawn, and BD having been produced towards E, let the angle $ADE=y$. Let $DZ=X$, the power to be applied at D, perpendicularly to BD, so as that it shall be upon the point of raising the weight Q. And, since we neglect the friction of the axle B, and the rigidity of the cord, we shall have $X.BD=Q.BT$.

Now, if DX be taken equal and contrary to DZ, it is manifest that DX will express the resistance, which the spindle opposes to the motion of the wheel. Let DX be resolved into the two DM, DN, the former perpendicular to AD, the latter in the direction of AD. And, because the power P turns the wheel, it has to overcome not only the force DM, tending to produce a rotation in a contrary direction, but also the friction $f.DN$, arising from the pressure DN exerted at D, by the spindle against

the tooth of the wheel, perpendicularly to the small arc described by the point D. Wherefore, it is necessary that

$$P \,.\, AK = (DM + f \,.\, DN)\, AD.$$

$$\text{But } DM = DX \cos.\, MDX = X \cos.\, y,$$

$$\text{and } DN = DX \sin.\, MDX = X \sin.\, y.$$

Wherefore, $P.\, AK = X \,.\, AD \,.\, (\cos.\, y + f \,.\, \sin.\, y)$;

and, substituting the value of X, from the first equation,

$$P. = \frac{Q \,.\, AD \,.\, BT}{AK \,.\, BD} (\cos.\, y + f \sin.\, y),$$

which was to be found.

725. *Coroll.* 1. From the point in which the tooth catches the spindle, to the point in which it leaves it, the point of contact D is continually changing, and the distances AD, BD, together with the angle y, go on increasing. Wherefore the power P is variable; and it may be of use to seek its greatest value.

The increase of BD is very small, and may be neglected, considering the small magnitude of the circle D, and its distance from the centre B. But, with respect to AD, we must take its greatest value; that is, the radius which goes from A to the point where the tooth leaves the spindle. And it remains for us to find the greatest value of the binomial $\cos.\, y + f \,.\, \sin.\, y$. Making its differential equal to nothing, this is found to be when $\tan.\, y = f$; in which case,

$$\cos.\, y + f \sin.\, y = \sqrt{(1 + f^2)}.$$

Wherefore, the greatest value of P, or that which will be ready to raise the weight, in the most difficult situation, will be

$$P = \frac{Q \,.\, AD \,.\, BT}{AK \,.\, BD} \sqrt{(1 + f^2)},$$

the semi-diameters BD, AD, being taken in the manner above specified.

726. *Coroll.* 2. If $f = \frac{1}{3}$, then $\sqrt{(1 + f^2)} = \frac{19}{18}$, nearly. Hence, the practical rule, delivered by Belidor, (*Arch. Hydr.*

tom. I. p. 104.) that when a weight is raised by the working of the teeth of a wheel against the staves of a trundle or lantern, in order to take into account the friction of the tooth, it is necessary to increase the weight, or else the arm of its lever, by an eighteenth part.

727. *Coroll.* 3. If the system is composed of several wheels, and of several pinions or spindles, and if a, a', a'', &c. are the arms of levers, on the side of the power, b, b', b'', &c. the arms of levers on the side of the weight, and if n be the number of junctions of a wheel with a pinion, which will always be less by one, than the number of pairs of the above-mentioned arms, then

$$P = \frac{b\,.\,b'\,.\,b''\,....}{a\,.\,a'\,.\,a''\,....}\,.\,(1+f^2)^{\frac{n}{2}}.$$

728. Scholium. We have considered only the resistance of the friction of the tooth, neglecting the frictions of the axles of the wheels, and the rigidity of the cord, which carries the weight Q. To obtain the exact value of P, we must also take into the account these resistances, which may be done in the manner explained in the preceding Chapter.

And here we may make use of the following very easy rule. In a system of toothed-wheels, which is a combination of several wheels and axles acting upon each other, for the value of the friction of the teeth, let all those arms of levers on the side of the weight, where the teeth of a wheel act upon those of a pinion be multiplied by $\sqrt{(1+f^2)}$; and in the rest, proceed in the manner mentioned in Art. 720, when combinations of several wheels and axles were treated of.

It is only requisite, that in the equation of Art. 720, the coefficient f shall represent the friction of the axles, which belongs to the *third* kind; whilst, in the multiplier $\sqrt{(1+f^2)}$, f represents the friction of the tooth, which belongs to the *first* kind. Hence, in forming the equations for each wheel, it is necessary to distinguish these coefficients by different marks, in order afterwards to assign to each the numerical value which belongs to it.

CHAP. XII.

ON THE EQUATION OF THE STATE BORDERING UPON MOTION IN THE PULLEY AND IN THE TAGLIA.

729. In the simple pulley, it is manifest that the equation of the state nearest to motion is the same as in the wheel and axle (720) making $a = b$. Passing on to the taglia, we shall facilitate the calculation, by supposing first, that all the branches of the string are parallel; secondly, that all the semi-diameters of the pullies, as also those of their axles, are equal to one another; thirdly, that f being very small, we may assume $\sqrt{(1+f^2)} = 1$; fourthly, that the rigidity of the cord is nearly proportional to the tension, so that we may make $\mu = 0$.

730. *Proposition.* If each of the semi-diameters of the pullies in the taglia $= a$, each of the semi-diameters of the axles $= b$, the number of branches of the cord drawing the moveable pullies $= K$, and if $\frac{a(\nu+1)+fr}{a-fr} = A$, the equation of the state nearest to motion will be

$$P = Q \cdot \frac{A^k (A-1)}{A^k - 1}.$$

Let the tensions of the branches of the cord be expressed by the letters t, t', t''.... as in Fig. 57; then between the two tensions t, t', we shall have (720) the equation

$$t'a = ta + (t'+t)fr + art,$$

whence $t' = At$. Similarly, it will be found that

$$t'' = At' = A^2 t; \; t''' = At'' = A^3 t, \text{ \&c.}$$

Hence, the last tension, or that of the cord which is immediately acted upon by the power, $= A^k t$, and the sum of all the others will be

$$t.(1 + A + A^2 + A^3 \ldots. + A^{k-1}),$$

$$\text{or } t.\frac{A^k - 1}{A - 1}.$$

Now the last tension is equal to the power P, and the sum of all the rest is equal to the weight Q.

$$\text{Wherefore, } P : Q :: A^k : \frac{A^k - 1}{A - 1},$$

$$\text{and } P = Q.\frac{A^k(A - 1)}{A^k - 1}.$$

731. Scholium. If the friction, and the rigidity of the cord vanish, then $A = 1$, which would give $P = Q.\frac{0}{0}$. But, the value of this fraction $\frac{0}{0}$ having been found by known methods, $P = \frac{Q}{k}$, as (698) it ought to be.

CHAP. XIII.

ON THE EQUATION OF THE STATE BORDERING UPON MOTION IN THE INCLINED PLANE, AND IN THE SCREW.

732. *Proposition I.* In the inclined plane, retaining the notation of Art. 702, the equation of the state nearest to motion is

$$P = Q \cdot \frac{\cos. m + f \sin. m}{\cos. n - f \sin. n}.$$

For, from the resolution of the forces P, Q, there results (702) a force, in a direction parallel to the plane, $= P \cos. n - Q \cos. m$, and a force perpendicular to the plane $= P \sin. n + Q \sin. m$; whence the friction $= f P \sin. n + f Q \sin. m$, therefore,

$$P \cdot \cos. n - Q \cdot m = f \cdot P \cdot \sin. n + f \cdot Q \cdot \sin. m,$$

$$\text{and } P = Q \cdot \frac{\cos. m + f \sin. m}{\cos. n - f \cdot \sin. n}.$$

733. *Coroll.* 1. When the power draws horizontally, we shall have $P = Q \cdot \frac{1 + f \cdot \tan. m}{\tan. m - f}$; and when the power acts in a direction parallel to the plane, $P = Q (\cos. m + f \sin. m)$.

These are the values which make the power to be upon the point of raising the weight. Those, which are sufficient simply to hinder the descent of the weight, are the same, the sign of f only being changed. This may be compared with Art. 579.

734. *Coroll.* 2. Here the most advantageous direction for the power, which is to raise the weight Q, is no longer that

which is parallel to the plane, but one which diverges from the plane, at an angle that has for its tangent f.

For, differentiating the value of P (732), making the angle n variable, and putting $dP = 0$, it is found that tan. $n = -f$.

But if the power is only to restrain the falling of the weight, we shall have tan. $n = f$; whence the direction of the power must converge to the plane at the same angle.

735. *Coroll.* 3. It often happens that the power drags the weight upon an inclined plane, by means of a rope, which passes over a fixt pulley: and then the equation of the state nearest to motion will be that of Art. 720, making (729) $b = a$, and, instead of Q, putting (732),

$$Q \cdot \frac{\cos. m + f \sin. m}{\cos. n - f \sin. n}.$$

Here, it is requisite that, in this formula, f shall represent the friction of the body sliding along the plane; but, in the equation (720) f must represent the friction of the axle of the pulley. These frictions are different, and therefore, in making the substitution, they must be distinguished by different marks.

736. *Coroll.* 4. If we suppose a wheel, having R for its radius, and r for the radius of its axle, to be placed on the plane, and the weight Q to rest upon it, then the power $P \cos. n - Q \cos. m$ tends to give a rotatory motion to the centre of the wheel about the point, where the wheel itself rests on the plane; and its momentum to produce this rotation is

$$= P \,.\, R \,.\cos. n - Q \,.\, R \,.\cos. m.$$

Now this rotation cannot take place, unless the wheel turns about its axis, to which is opposed the friction

$$f \,.\, P \,.\sin. n + f \,.\, Q \,.\sin. m,$$

between the axle and the circle which embraces it; and the momentum of this resistance is

$$= f \,.\, P \,.\, r \,.\sin. n + f \,.\, Q \,.\, r \,.\sin. m.$$

Wherefore, in the state nearest to motion, we shall have

$$P.R.\cos.n - Q.R.\cos.m = f.P.R.\sin.n + f.Qr.\sin.m,$$

$$\text{whence, } P = Q.\frac{R.\cos.m + fr.\sin.m}{R.\cos.n - fr.\sin.n}.$$

In this case, the most advantageous direction for a power, which is to raise a weight, will be $\tan.n = -\frac{fr}{R}$.

The conclusion is the same, whether the weight be carried by a single wheel, or by several equal wheels; because the pressure of the weight being distributed amongst the wheels, the sum of the friction will always be

$$fP\sin.n + fQ\sin.m,$$

and the sum of their momentums will be

$$fP.r.\sin.n + f.Qr.\sin.m,$$

as before.

Hence, it may be seen in what manner wheels aid a power, which draws a weight up a plane; and the advantage will appear to be still greater, if it is considered that, in this case, the friction is of the *third* kind, and that the coefficient f has a value less than that which it has, when the weight is dragged along the ground.

737. *Coroll.* 5. When the plane is horizontal and the direction of the draught is also horizontal, then $P = \frac{frQ}{R}$. If the power dragged the weight on the ground, we should have $P = fQ$, and the power would have to overcome the whole friction arising from the pressure of the weight Q. But, by the aid of wheels, first, the friction is less, it being changed, from what it would have been in a sliding motion, to the friction of the axles; secondly, of this friction itself the power has only to overcome such a part, as the radius of the axle is of the radius of the wheel.

Let, for example, a weight of 48950 chilograms, be to be drawn over an horizontal plane. Let the weight be placed upon a carriage with four wheels, in each of which let the radius of

the axle be to the radius of the wheel, as two to nine. For the friction of the axles, let us suppose (517)

$$f = \tfrac{1}{7}\text{; and, since } \frac{r}{R} = \frac{2}{9},$$

we shall have

$$P = \frac{2}{63}\,Q = 1553.97 \text{ chilograms.}$$

738. *Proposition II.* In the screw, retaining the notation of Art. 707, the equation of the state nearest to motion is

$$P = Q \,.\, \frac{b}{a} \,.\, \frac{h + 2 f \pi b}{2 \pi b - f h}.$$

Following the same method as in Art. 707, we shall still have

$$P : X :: 2 \pi b : 2 \pi a.$$

Then comparing together the forces X and Q, we shall have (733)

$$X : Q :: 1 + f \,.\, \tan . m : \tan . m - f,$$

$$\text{or, since (705) } \tan . m = \frac{2 \pi b}{h},$$

$$X : Q :: h + 2 f \pi b : 2 \pi b - f h;$$

and multiplying this by the preceding proportion, we shall thence obtain the equation stated in the proposition.

CHAP. XIV.

ON THE EQUATION OF THE STATE BORDERING UPON MOTION IN OTHER COMPOUND MACHINES.

739. By the help of the rules delivered in the preceding chapters, there is no combination of machines, in which we cannot readily find the equation of the state nearest to motion. It would be superfluous to extend our investigations to many particular instances. Let it suffice to have given an example, in the machine described by Belidor (*Archit. Hydr.* tom. I. p. 122.) under the name of *Levier de la Garousse*.

740. The weight V (Fig. 67.) placed on a carriage moveable upon four wheels R, is drawn by means of a rope $TZSN$ wrapped over the moveable pulley ZS. The end T of the rope is fastened to an immoveable support; the other end is attached to the cylinder NH, which turns round, together with the wheel OGF, from G towards F. Motion is communicated to the wheel, in the following manner. AB is a beam moveable about the fulcrum C. On each side of the fulcrum, at D, E, are two eyelets, in which move freely the crooks DG, EF, that terminate in the periphery of the wheel, and take hold, by one at a time, of the teeth with which that periphery is furnished; as is shewn in the figure. When the beam AB is horizontal, the straight lines GI, FK, joining each of the eyelets and the point where the crook touches the periphery, are tangents to the periphery at G, F. If, now, at the extremity A, there be applied a vertical power P, which will make the beam turn about the fulcrum C, the eyelet E will be raised, and the crook EF will urge on the point F of the wheel. If, after this, another power P' is applied at the extremity B, which will cause the beam to turn in a contrary direction, the eyelet D will be raised, and the

crook DG will urge on the point G. Thus, by the alternate action of the two powers P, P', the wheel will be made continually to turn in the same direction GF, and the weight will be continually urged on.

741. Let the weight $V = 48950$ chilograms; let the radius of the axles of the wheels R be $\frac{2}{9}$ of the radius of the wheels themselves; let the radius of the pulley $ZS = 0.108$ metres, and let the radius of its axle be the tenth part of it, that is, 0.011 metres nearly. Let the diameter of the rope $TZSN = 0.037$ metres; the radius HN of the cylinder $= 0.1355$ metres; the radius HF of the wheel $= 0.65$ metres; the radius of the axle, upon which the wheel turns, $= 0.027$ metres. Let $AC = CB = 2.761$ metres, and let it be supposed that, when AB is horizontal, and when from C there have been drawn the perpendiculars CL, CM, to the directions FK, GI, each of these perpendiculars is the tenth part of AC, that is, 0.276 metres. Lastly, let the beam weigh 97.9 chilograms; and let the radius of the small axle C, upon which it turns, $= 0.02$ metres.

These measures being given, it is required to find the value of the power P, which, drawing at A the horizontal beam AB, is upon the point of movingit; causing, at the same time, the wheel to revolve, and urging forwards the weight V.

742. First, then, on account of the wheels R, if we put $f = \frac{1}{7}$, the force necessary to put in motion the weight is reduced to 1553.97 chilograms, as is deduced (737) from the formula $P = \dfrac{frQ}{R}$.

Next, considering the action of the pulley ZS, since the two ropes TZ, NS, are nearly parallel, the tension of each of them will be the half of that weight, that is, 776.985 chilograms. Since, however, the pulley intervenes, the tension of the rope TZ will indeed be equal to a weight of 776.985 chilograms, which we shall call Q; but the tension of the other rope NS, which we shall call Y, will be greater than Q, by the addition caused by the friction of the pulley, and the rigidity of the cord.

Let the radius of the pulley $= h$, and the radius of its axle $= r$. We shall have between the forces Y, Q, the equation

$$(a)\ Yh = Qh + (Y + Q)fr + h(\mu + \nu Q).$$

743. Let us now consider the wheel and the cylinder. Let the radius of the wheel $HF = a$, the radius of the cylinder $HN = b$, the radius of their common axis $= r'$. Let X be the force, which the hook EF exerts at F against the tooth of the wheel, acting in the direction of the tangent FK, and let the angle $AEF = \theta$. By means of this wheel and axle, the two forces X, Y, are opposed to each other; and, since the angle of their directions is $= \theta$, we shall have (718) the equation

$$(b)\ Xa = Yb + fr' \sqrt{(X^2 + 2\,XY\cos.\theta + Y^2)} + b(\mu + \nu Y)$$

744. Lastly, we have to consider the lever AB. Put $AC = m$, $CL = m$, the radius of the axle $C = r''$. The directions of the forces P, X, applied at the extremities of the lever, make with one another an angle $= 90^\circ - \theta$. Hence, if Π be the weight of AB, which presses wholly on the point C, the pressure of the axle C will be

$$\sqrt{\{(P + \Pi)^2 + 2(P + \Pi)\sin.\theta + X^2\}}.$$

Wherefore, we shall have the equation

$$(c)\ Pm = Xn + fr'' \sqrt{\{(P + \Pi)^2 + 2(P + \Pi)\sin.\theta + X^2\}}.$$

Now, from the three equations (a), (b), (c), determining successively Y, X, P, we shall have at length the power P expressed in terms of Q, the value of which we know to be $= 776.985$ chilograms.

745. The equations may be rendered much more simple, by neglecting some elements, which affect but little the final value. We may omit the coefficient μ, which, in reality, is very small, in comparison of νQ, νY, on account of the great magnitude of the weights Q and Y. The forces X and Y may also be considered as parallel, as may also the forces P, X; whence the pressures upon the axles H, C, will be respectively $X + Y$, $P + \Pi + X$. In so doing, we only make these pressures a little greater than they are in reality; but the difference thence arising, in the momentum of the friction of the axles, is very

small, on account of the smallness of the coefficients fr', fr''. With these simplifications, the equations become

$$Yh = Qh + (Y + Q)fr + h\nu Q,$$
$$Xa = Yb + (X + Y)fr' + b\nu Y,$$
$$Pm = Xn + (P + \Pi + X)fr'';$$

whence is deduced

$$P = Q.\frac{n+fr''}{m-fr''}.\frac{b(\nu+1)+fr'}{a-fr'}.\frac{h(\nu+1)+fr}{h-fr} + \Pi.\frac{fr''}{m-fr''}.$$

746. In substituting the numerical values (741), it must not be forgotten to increase the semi-diameters h, b, of the pulley and of the cylinder, by adding to them the semi-diameter of the rope. The coefficient f of friction is supposed to be $\frac{1}{7}$; the coefficient ν of the rigidity of the rope is determined from the data, by Art. 528; and it is found that $\nu = 0.088$. When the calculation is completed, we have

$$P = 0.03 \times Q + 0.1 = 23.41 \text{ chilograms.}$$

Belidor finds $P = 60$ chilograms nearly. The variance is principally owing to his having estimated every kind of friction, without distinction, at a third part of the pressure.

SECTION III.

ON MACHINES IN MOTION.

CHAP. XV.

ON THE UNIFORMLY ACCELERATED MOTION OF MACHINES.

747. IF the power be increased beyond the limit of the state nearest to motion, it raises the weight, with a motion of a kind differing according to the different nature both of the power, and of the resistances, which oppose the movement of the machine. And, because the motion is continued, it is necessary, on account of the resistances, that the power shall be a force constantly applied, that is (203) an accelerating force.

If this force be constant, if the resistance also be constant; and if their momentums be always the same, the machine will turn round, and raise the weight with an uniformly accelerated motion. But if, during the progress of the motion, either the forces, or their momentums, suffer an alteration, the motion will still be accelerated, but not uniformly. Lastly, it may be, that, the moving force continually decreasing, or the resistance continually increasing, these opposite forces may become equal to one another: in which case the acceleration will cease altogether, and the motion of the machine will become equable; the machine preserving that degree of velocity, which it had acquired at the end of the acceleration. We shall consider these three cases, in the three following chapters.

748. Let us first suppose a weight to be raised by means of a wheel and axle. Let the power, which moves the machine, be another weight P, hanging by a string wrapped about the periphery of the wheel. In this case, P and Q being constant, and their momentums being invariable, the motion must be uniformly accelerated.

749. *Proposition.* If P be the weight moving a wheel and axle, Q the weight raised by it; a, b, the respective semi-diameters, or arms of the levers; and S the momentum of inertia of the machine referred to the axis of rotation; the weight P will descend, and the weight Q will rise, with an uniformly accelerated motion; and, if ϕ, ϕ', be put for the respective accelerating forces, then

$$\phi = ag \cdot \frac{Pa - Qb}{gS + Pa^2 + Qb^2}; \quad \phi' = bg \cdot \frac{Pa - Qb}{gS + Pa^2 + Qb^2}.$$

For, let ω be the angular velocity of the system; then (341) will $\frac{d\omega}{dt}$ be equal to the momentum of the impelling force divided by the momentum of inertia. Now, $P.a$ being the momentum of the moving force P, and $Q.b$ the momentum of the weight Q, $Pa - Qb$ will be the whole momentum of the impelling force. With respect to the momentum of inertia, since the masses $\frac{P}{g}$, $\frac{Q}{g}$, at every instant, move with the velocities with which the extremities of the respective semi-diameters a, b, revolve, they must be understood to be collected in the extremities of these semi-diameters, whence their momentums of inertia will be

$$\frac{Pa^2}{g}, \quad \frac{Qb^2}{g}; \quad \text{and} \quad S + \frac{Pa^2}{g} + \frac{Qb^2}{g}$$

will be the momentum of inertia of the whole system. Wherefore,

$$\frac{d\omega}{dt} = g \cdot \frac{Pa - Qb}{gS + Pa^2 + Qb^2} = g \cdot M,$$

$$M \text{ being put for } \frac{Pa - Qb}{gS + Pa^2 + Qb^2}.$$

Wherefore, the angular velocity $\omega = g\int M dt$: and if u, u', be put for the velocities at the extremities of the semi-diameters a, b, which are the velocities of the weights P, Q, we shall have (338),

$$u = a\omega = ag\int Mdt; \quad u' = b\omega = bg\int Mdt;$$

whence (206)

$$\phi = \frac{du}{dt} = agM; \quad \phi' = \frac{du'}{dt} = bgM.$$

750. *Coroll.* 1. If it be required to find the value of the radius a, so that the weight may be raised with the greatest velocity, this will be had from the equation

$$d \cdot \frac{Pa - Qb}{gS + Pa^2 + Qb^2} = 0,$$

differentiating upon the supposition that a is variable. And since, when a varies, S also varies, before differentiating it will be necessary to substitute for S its value in terms of a. We may, however, dispense with this, whenever S may be neglected in comparison with the other terms of the denominator; or whenever S may be taken for a constant quantity, in consequence of its varying very little with the variation of a.

751. *Coroll.* 2. In the fixt pulley, $a = b$, and $2T$ being put for the weight of the pulley, we have (311) $gS = Ta^2$. So that the weight P will descend, and the weight Q will rise, each with an equal accelerating force, which will be

$$g \cdot \frac{P - Q}{T + P + Q};$$

and hence the use of Atwood's machine, for verifying the laws of motion.

752. *Coroll.* 3. If the weight Q does not rise vertically, but through an inclined plane, the preceding formulas will still serve; only that, in the numerators, instead of Q we must put (702) $Q \cdot \frac{\cos. m}{\cos. n}$.

753. *Coroll.* 4. Lastly, if the resistauces of friction, and of the string, are to be taken into the account, the formulas of Art. 749. will still serve; except that, in the numerator, instead of Qb, we must put (720)

$$Qb + (P + Q)fr + b(\mu + \nu Q).$$

But if the weight Q be drawn up an inclined plane, after the above-mentioned substitution, it will be necessary, in the numerator, to change Q into

$$Q \cdot \frac{\cos. m + f \sin. m}{\cos. n - f \sin. n};$$

regard being had to what has already been said (735) respecting the values of the coefficient f.

As the coefficients of the resistances f, μ, ν, may (511, 520, 524) be considered as constant, it is clear that the motion remains uniformly accelerated.

CHAP. XVI.

ON THE VARIABLY ACCELERATED MOTION OF MACHINES.

754. WHEN it is intended, by the continued turning of a wheel, to produce an alternate motion, a crank is frequently made use of. This is most commonly the case in wheels employed for the purpose of raising the pistons of hydraulic pipes. The rod of the piston is attached to the elbow F (Fig. 68.) of the crank. As the wheel turns round, the point F rises to R, describing the semi-circle FGR; it afterwards returns to F, through the opposite semi-circle RTF; and thus the piston rises and sinks by turns.

In this movement, even when the moving force is equivalent to a constant weight P, and when also the force of the piston is equivalent to a constant weight Q, hanging from the arm of the crank, the motion cannot be uniformly accelerated. Because, whilst the piston ascends through the semi-circle FGR, the arm of the lever of the weight Q is continually changing; it is nothing at the point F, it is greatest at the point G, where it is equal to the breadth CF of the crank, and it vanishes at the highest point R. Whence, it is easily seen that the motion will be continually accelerated, but by less and less degrees, in the first quadrant, FG, and then by greater and greater degrees, in the second quadrant GR. Let us proceed to determine the laws of this movement.

755. *Proposition.* When a weight is raised by means of a crank, (Fig. 68.) to determine the velocity in any assigned point of the semi-circle FGR.

Let a be the arm of the lever, at which the moving force P acts, let the breadth of the crank $CF = b$, the momentum of

inertia of the machine $= S$, including therein the weights P, Q; lastly, let ω be the angular velocity of the machine, which being known, the velocity of each point will be known. At the end of the time t, let the point F have arrived at M, having described the angle $FCM = \psi$; then will the momentum of the weight $= Qb \sin. \psi$. Hence (341)

$$\frac{d\omega}{dt} = \frac{Pa - Qb \sin. \psi}{S},$$

or, since $\omega = \dfrac{d\psi}{dt}$,

$$\frac{dd\psi}{dt^2} = \frac{Pa - Qb \sin. \psi}{S}.$$

Multiplying by $2\,d\psi$, and integrating so that $\psi = 0$ may give $\dfrac{d\psi}{dt} = 0$, we shall have

$$\left(\frac{d\psi}{dt}\right)^2 = \omega^2 = \frac{2\,Pa\psi - 2\,Qb\,(1 - \cos. \psi)}{S}.$$

756. *Coroll.* 1. Hence are deduced the following relations between the spaces and the squares of the velocities,

$$\begin{aligned}
\psi &= 0, \ldots\ldots\ldots S\omega^2 = 0;\\
\psi &= \tfrac{1}{4}\pi \ldots\ldots\, S\omega^2 = \tfrac{1}{2}\,Pa\pi - 0.586\,Qb;\\
\psi &= \tfrac{1}{2}\pi \ldots\ldots\, S\omega^2 = \quad Pa\pi - 2\,Qb;\\
\psi &= \tfrac{3}{4}\pi \ldots\ldots\, S\omega^2 = \tfrac{3}{2}\,Pa\pi - 3.414\,Qb;\\
\psi &= \quad \pi \ldots\ldots\, S\omega^2 = 2\,Pa\pi - 4\,Qb;
\end{aligned}$$

from which it is easy to perceive that the motion through FGR is not uniformly accelerated, but always less in the quadrant FG, and always more in the quadrant GR.

757. *Coroll.* 2. Further, considering the entire ascent from F to R, it appears that the squares of the velocities acquired at the points G, R, are to one another as the spaces described FG, FR; exactly as if the motion were uniformly accelerated. Whence, it is evident, that, as the anomalies of the motion in the intermediate points sufficiently compensate each other, we may consider the acceleration as if it were uniform, through the

whole course of the weight's ascent. The constant accelerating force, which is equivalent to the actual variable force, will be obtained (210) by dividing the square of the velocity by the double of the space, and will be $= \dfrac{Pa - \dfrac{2Qb}{\pi}}{S}$. Hence, the mean arm of the lever will be $= \dfrac{2b}{\pi} = \dfrac{7}{11}b$; and hence the following rule.

758. *Coroll.* 3. "When a weight is raised by means of a crank, the arm of the lever, at the end of which it acts, may be considered as constant, and equal to $\frac{7}{11}$ of the breadth of the crank."

759. SCHOLIUM 1. The ascent having been completed, the descent follows through the arc RTF, which brings back the piston to the point, whence it set out. In this descent, the moving power P does not at all oppose the weight Q. When the crank is employed to raise the pistons of tubes, the piston, after having reached the highest point R, descends by its own weight, and does not exert any force on the machine. Hence, during the whole of the time employed by the crank in returning, and in bringing back the elbow to the lowest point, the moving force is idle.

760. SCHOLIUM 2. To avoid this loss of time, a double crank has been contrived. Two equal cranks are constructed, on the same axle, and situated in the same plane, but turned towards opposite parts. To their elbows E, F, are attached the rods of the pistons of two equal tubes, so that whilst the one ascends through its semi-circle, the other descends through the opposite semi-circle.

Since the descending piston exerts no force, it is evident that there is no other difference between the simple and the double crank, except that the former is interruptedly, and the latter continually, applied, to raise the weight Q through the arc FGR, in the manner above explained; whence the arm, at which the weight itself acts, may be considered as constant and $= \frac{7}{11}CF$.

Combinations of three or four cranks, variously disposed on the same axle, are also used; the theory of which will be easily understood, by proceeding after the manner of the foregoing calculation: and thus much let it suffice to have said of this machine, for the purpose of giving an example of a motion not uniformly accelerated, and of the method of equalizing its irregularities.

CHAP. XVII.

ON THE UNIFORM MOTION OF MACHINES.

761. The acceleration of a machine slackens, and its motion has a tendency to become uniform, whenever, in the process of the motion, either the accelerating force decreases, or the resistance increases. We have an example of the first case, in machines moved by animal strength, the which force (464) goes on decreasing, in proportion as the motion of the animal is quickened. An example of the second case occurs in machines furnished with a fly, which, striking the air with its limbs, meets with a greater resistance, the greater (226) is the velocity.

762. *Proposition.* The same supposition being made as in Art. 750, if, instead of the constant weight P, a force F moves the machine, which is any function of the velocity, the weight Q will rise with an accelerated motion, and the accelerating force will be

$$\phi' = b \cdot \frac{Fa - Qb}{S + Fa^2 + Qb^2}.$$

The demonstration is the same as that of Art. 750.

763. *Coroll.* 1. The force ϕ' being known, the well known equations $\phi' dt = du$, $\phi' ds = n\,du$, give the increments of the velocity, and all the consequences of the motion. And if F decrease as the velocity increases, the accelerating force ϕ' will also go on decreasing; whence it appears, that the motion is accelerated at its beginning, and afterwards goes on tending to become uniform.

764. *Coroll.* 2. The motion becomes uniform, as soon as

$$\phi' = 0, \text{ or } Fa = Qb.$$

Whence it appears, that the force required in the moving power to keep the machine in a permanent state of equable motion, is the same which is required to keep it in equilibrium; which is evident also of itself.

765. *Coroll.* 3. The same equation $Fa = Qb$ will also give us the equable and permanent velocity of the machine. For an example of this, let F be supposed to be the force of a man, expressed by some one of the three formulas already (486) proposed.

The value of F having been substituted in the equation $Fa = Qb$, we shall thence have, according to the different hypotheses,

$$1.\ v = h\left(1 - \frac{bQ}{ag}\right).$$

$$2.\ v = h\sqrt{\left(1 - \frac{bQ}{ag}\right)}.$$

$$3.\ v = h\left(1 - \sqrt{\frac{bQ}{ag}}\right).$$

This will be the equable velocity of the point of application of the moving power F; and, multiplying it by $\frac{b}{a}$, we shall have the velocity with which the weight Q is raised.

766. *Coroll.* 4. If the friction, and the rigidity of the cord, were to be taken into the account, we must proceed as in Art. 753; whence, the equation of the permanent state will be

$$Fa = Qb + (F + Q)fr + b(\mu + \nu Q),$$

and if the machine does not lift the whole weight of the mass Q, but draws it along a plane, instead of Q, we must make the substitution there (753) indicated.

767. *Coroll.* 5. In general, if Nk be the sum of the

momentums of the resistances, the equation of equable motion will be

$$Fa = Qb + Nk;$$

where, if F, and N, are functions of the velocity, it will be proper to substitute their values, expressed by the velocity of a determinate part of the machine; and then the equation itself will give us the velocity, which belongs to the equable and permanent rotation of the machine.

CHAP. XVIII.

ON THE MOST ADVANTAGEOUS DISPOSITION OF MACHINES.

768. In machines, which raise a weight by an equable motion, *the effect of the machine* is measured by the product of the weight raised, multiplied by its velocity. Similarly, *the effect of the moving force* is measured (475) by the product of the force itself multiplied by its velocity; that is, by the product of a weight, equivalent to the force exerted by the moving power, multiplied by the velocity with which it proceeds.

769. *Proposition I.* If, by means of a wheel and axle, a moving power F, proceeding with an uniform velocity v, raise a weight Q, with a velocity u, the effect of the machine will be equal to that of the force.

For, on account of the uniform motion (764) $Fa = Qb$. But

$$a : b :: v : u;$$

wherefore, $Fv = Qu$; and Qu is the effect of the machine, Fv the effect of the force.

770. Scholium. It appears that the power is to the weight, as the velocity of the weight is to that of the power. This proportion obtains as well in equable motion, as in the state of equilibrium; except that, in motion, the actual velocities are compared together, and in equilibrium the virtual velocities. And it obtains, not only in the wheel and axle, but in every imaginable machine.

771. *Coroll.* 1. The effect of a force, applied to carry a weight with an uniform motion, is not at all increased, whatever be the machine that is employed. The common opinion, then, that the machine augments or multiplies the force, is groundless. In what the utility of machines really consists will be explained further on.

772. *Coroll.* 2. If the resistances are to be taken into the account, the equation of the permanent state being, (767)

$$Fa = Qb + Nk,$$

$$\text{or } Fv = Qu + \frac{Nku}{b},$$

it is found, that the effect of the machine Qu is not equal to the effect of the force Fv. Hence, so far is the machine from having a power to augment the force, that it even diminishes it; and the more, in proportion as it is more subject to friction, and to the other resistances.

773. *Coroll.* 3. When the force F decreases as the velocity increases, there is (488) such a value of the force, as renders its effect Fv a maximum. This same value will, therefore, make the effect of the machine also a maximum. And hence originates the enquiry, how to dispose the machine to the greatest advantage.

774. *Proposition II.* There being given the weight Q to be raised equably by a force F, which is a given function of the velocity, to determine the ratio $\frac{b}{a}$, of the radius of the axle to that of the wheel, so that the effect obtained may be a maximum; and *vice versâ.*

First, by the equation (488) $d.Fv = 0$, the value of F must be sought, proper for the greatest effect. Then, this value must be substituted in the equation $Fa = Qb$, whence, either of the two elements Q, or $\frac{b}{a}$, being given, the other may be found.

But if resistances intervene, the equation $Fa = Qb + Nk$ must be used.

775. *Coroll.* 1. Let F be the force of a man; its value, for the greatest effect, will be (488) one of the three $\frac{1}{2}g$, $\frac{2}{3}g$, $\frac{4}{9}g$. And hence the problem will be solved by one of these three equations,

$$ga = 2Qb; \; 2ga = 3Qb; \; 4ga = 9Qb,$$

according to the hypothesis (486) that is adopted.

776. *Coroll.* 2. If not one agent only, endowed with the force F, but several such agents, in number n, were applied to the wheel, the equation would be $n\,Fa = Qb$. And, here, of the three elements n, Q, $\frac{b}{a}$, two being given, the third might be determined for the most advantageous effect.

777. *Coroll.* 3. The utility of this theory seems to require, that it should be illustrated by an example.

Let us assume, then, for the permanent force of a man, the expression

$$F = g\left(1 - \frac{v}{h}\right)^2; \text{ and let } g = \overset{\text{chil.}}{21}, \; h = \overset{\text{metres.}}{1\,.\,95};$$

and with a crane, where the radius of the wheel is twelve times as great as that of the axle, a mass, weighing 2800 chilograms, being to be raised, let it be required to find what number (n) of labourers it will be best to employ in turning the wheel.

The equation $4\,nga = 9\,Qb$ will give $n = 25$.

It is easy to satisfy ourselves that this is in reality the most advantageous number. For, making $n = 25$, we have

$$F = \frac{Qb}{na}, = 9\,.\,33; \text{ and } v = h\left(1 - \sqrt{\frac{F}{g}}\right) = 0.65.$$

Wherefore, the effect of each will be $= 6.066$; and the effect of the machine $= 25 \times 6.066 = 151.65$.

Let now any other number of workmen be tried; the effect of each will always turn out to be less than 6.066; and the effect of the machine will have always a less proportion to the number of the labourers. If, for example, the number of labourers be increased to 100, it will be found that the effect of each does not reach the half of the preceding effect. And thus by quadrupling the agents, it does not happen that the effect of the machine is doubled.

CHAP. XIX.

ON THE REAL ADVANTAGES OF MACHINES.

778. The false opinion, which persons, unskilled in the nature and the power of machines, are apt to conceive, often encourages empty errors, and mischievous deceptions. One of the most common of these conceits is that of considering machines, as available to increase and multiply the force of agents; which is not always true. To form a just notion of the aid, which may be expected from machines, looking to the uses to which they are most commonly put, we shall divide them into two classes; those intended simply to sustain a weight, and those intended to draw it, or to raise it, equably.

779. In machines of the first class, both the effect of the machine and the immediate effect of the power, can only be estimated by the weight sustained.

This being understood, it is evident that the machine increases the effect of the power; so that, for example, a force of 10 chilograms will sustain, by means of a lever, 100 chilograms; provided that the arm of the force be ten times as long as that of the weight.

780. If it be asked, how the force can ever produce an effect so much greater than itself, we shall perceive, if we consider well, that the force 10 does not really sustain the whole weight 100, but only the tenth part of it. Let the lever be supposed to be of the second kind; the force 100 may be resolved into two, the one equal to 90, which acts upon the fulcrum, and the other equal to 10, which acts at the point of application of the power. The first is entirely sustained by the prop, and the power sustains the second alone. Archimedes required only a fixt point, to hold the terraqueous globe in equilibrium. If he had found it, says Carnot, it would not, in

reality, have been Archimedes, but the fixt point, which would have sustained the earth.

781. In machines of the second class, neither the effect of the machine, nor that of the power, can be estimated simply by the weight raised; otherwise the measure of the effect would be altogether vague and indeterminate. In fact, any force, however small, may carry a weight of any assignable magnitude, however great; if only it be granted, that the weight admits of being divided, and of being carried one piece at a time. Wherefore, it is necessary to take into account the time, also, in which the power can carry the weight through a given space, or the velocity with which the weight is carried; and on this account it is, that the effect (768) is measured by the product of the weight multiplied by the velocity.

Now, upon this principle, we have already shewn, that the machine does not increase the effect of the force. If a man, with a force equivalent to 10 chilograms, raise, by means of a machine, a weight of 100 chilograms, he moves with a velocity ten times as great as that of the weight, and does as much, as if, operating without any machine, he carried those 100 chilograms at ten journies, loading himself with 10 chilograms at a time. In a word, what is gained in the quantity of the weight moved, is lost in the velocity; and the effect remains the same.

782. Between the two classes of machines above described, there is, then, this characteristic difference, that the first add to the effect of the power, the second do not add to it.

There is another difference, not less remarkable, respecting the resistances of friction, and of ropes, and other resistances. In machines of the first class, these resistances are all of them advantageous to the power, and themselves also sustain their portion of the weight; whence there remains so much the less of it for the power to support. On the contrary, in machines of the second class, the resistances are all of them detrimental to the power, and form part of the weight to be overcome; whence, on this account, a force is required greater than that which would be required, in the immediate application of the power.

783. These things being understood, we may now easily bring in review the true scope, and the real utility of machines. Machines of the first class seem to increase the effect of the power; and they do this by conveniently distributing the weight between the power and the prop.

Machines of the second class serve, not to augment the effect of the power in quantity, but to modify its quality, as is wanted; and they do this in the following manner. The effect of the machine being the product of the weight multiplied by its velocity, we can increase, at pleasure, one of the two factors, provided that the other is proportiouably diminished. Thus, by means of a machine, we can move a weight enormously great, provided that we are content to move it slowly; or, *vice versâ*, we can move a weight with very great velocity, provided that it is a small weight; whereas, by the immediate application of the force, we can hardly go beyond certain limits, either of velocity or of weight.

784. Another use of machines is that of enabling us to apply to the carrying, or to the sustaining of weights, those forces, which are not immediately applicable to such a purpose, and with that direction, which is most convenient for the power. The impact of a stream, or the force of a beast, could not be used in drawing water from a well, or stones from a quarry, if it were not for the intervention of a machine; and man himself draws water more easily by means of a pulley, than he would do by drawing the bucket vertically upwards.

785. Although the choice of a machine for a given purpose, and its most convenient disposition, cannot be reduced to precept, it may yet be of use to give some few general hints.

1st. In machines intended for the carriage of weights, it is best to reduce, to the least possible quantity, the friction, and the other resistances; but not so in those destined simply to support weights. Wherefore, generally speaking, in the first case, simple machines are preferable, in the second compound machines.

2dly. When the force decreases as the velocity increases, it is best to dispose the machine, so that the force of each agent

may be exerted to the greatest advantage; which will be done in the manner above explained (774).

3dly. In general, it is proper to take care, that the whole of the force is employed in producing the intended effect; and that no part of it is wasted on effects foreign to the true purpose of the machine. Thus, in the lever, if the power act obliquely to its arm, a part of it will be lost, in producing a stress against the point of support, with an useless consumption of force, to the mere detriment of the machine.

APPENDIX.

ON THE PRINCIPLE OF VIRTUAL VELOCITIES, AND ITS USES IN MECHANICS.

CHAP. I.

EXPLANATION AND DEMONSTRATION OF THE PRINCIPLE OF VIRTUAL VELOCITIES.

786. To the point A (Fig. 69.) let there be applied a force S, which acts in the direction of the straight line AB. Imagine the point A to be carried to another point a, very near to A. If from a there be drawn ab perpendicular to AB, Ab will be the small space described by the point A, in the direction AB, which is the direction of the force S. By the *Virtual Velocity*, then, of a force, is meant the indefinitely small space, which, in a given indefinitely small motion, is described by the point of application of the force, in the direction of the force itself.

787. The product $S.Ab$, that is, the product of the force multiplied by its virtual velocity, is here called the *Momentum* of the force S.

These terms are adopted for the sake of abridgement. But it is necessary to guard against confounding this

meaning of the word momentum, with other different quantities, which, in other parts of the work (54.103) have been designated by the same name.

788. *Proposition I.* If several forces act upon the same point, the momentum of the resultant is equal to the sum of the momentums of the components.

Let the forces S, S', S'', &c. represented in quantity and direction, (Fig. 69.) by the straight lines

$$AB = s,\ AC = s',\ AD = s'',\ \&c.$$

meet in the point A, and let the straight line $AR = u$ be their resultant. When this system is referred to three rectangular axes, let x, y, z, be the co-ordinates of the point A. Imagine this point to be transferred to a, describing in the directions of the three co-ordinates the indefinitely small spaces dx, dy, dz. And, considering the straight lines $s, s', s'' \ldots\ldots u$, as functions of x, y, z, it is plain that the indefinitely small spaces Ab, Ac, $Ad \ldots\ldots Av$, described by the point A in the directions $s, s', s'' \ldots\ldots u$, will be respectively

$$ds,\ ds',\ ds'' \ldots\ldots\ldots du:$$

and the momentums of the forces will be

$$Sds,\ S'ds',\ S''ds'' \ldots\ldots\ldots Vdu.$$

Let, now, a, b, c, be the co-ordinates of the point B. Then,

$$AB = s = \sqrt{\{(x-a)^2 + (y-b)^2 + (z-c)^2\}};$$

whence

$$\left(\frac{ds}{dx}\right) = \frac{x-a}{s};\ \left(\frac{ds}{dy}\right) = \frac{y-b}{s};\ \left(\frac{ds}{dz}\right) = \frac{z-c}{s}.$$

Resolve the force S into three forces parallel to the co-ordinates x, y, z. These will be

$$S.\frac{x-a}{s};\ S.\frac{y-b}{s};\ S.\frac{z-c}{s};$$

$$\text{or, } S.\left(\frac{ds}{dx}\right);\ S.\left(\frac{ds}{dy}\right);\ S.\left(\frac{ds}{dz}\right).$$

Let the same resolution be made for the other forces S', S'', &c., and also for the force V, which, in like manner, will be decompounded into the three

$$V.\left(\frac{du}{dx}\right);\ V.\left(\frac{du}{dy}\right);\ V.\left(\frac{du}{dz}\right):$$

and we shall have (30) the equations

$$V.\left(\frac{du}{dx}\right) = S.\left(\frac{ds}{dx}\right) + S'.\left(\frac{ds'}{dx}\right) + S''.\left(\frac{ds''}{dx}\right)\ldots.$$

$$V.\left(\frac{du}{dy}\right) = S.\left(\frac{ds}{dy}\right) + S'.\left(\frac{ds'}{dy}\right) + S''.\left(\frac{ds''}{dy}\right)\ldots.$$

$$V.\left(\frac{du}{dz}\right) = S.\left(\frac{ds}{dz}\right) + S'.\left(\frac{ds'}{dz}\right) + S''.\left(\frac{ds''}{dz}\right)\ldots.$$

which multiplied respectively by dx, dy, dz, and then added together, give

$$V.du = S.ds + S'.ds' + S''ds''\ldots\ldots$$

789. SCHOLIUM 1. The proposition is true, even when the point A, coming to a, describes a finite space. Because, when the point A, describes, in the directions x, y, z, the finite spaces Δx, Δy, Δz, the variations of the straight line s, corresponding to these three spaces, will be

$$\frac{x-a}{s}.\Delta x,\ \frac{y-b}{s}.\Delta y,\ \frac{z-c}{s}.\Delta z;\ \text{whence}$$

$$\left(\frac{\Delta s}{\Delta x}\right) = \frac{x-a}{s};\ \left(\frac{\Delta s}{\Delta y}\right) = \frac{y-b}{s};\ \left(\frac{\Delta s}{\Delta z}\right) = \frac{z-c}{s}.$$

From which we shall have, in the same manner as before,

$$V\Delta u = S\Delta s + S'\Delta s' + S''\Delta s''\ldots\ldots..$$

790. Scholium 2. The proof of the preceding proposition may also be deduced from the most simple elementary notions.

For let there be two forces $AB = S$, $AC = S'$ (Fig. 70.) acting upon the point A, and let $AR = V$, be their resultant; which will be the diagonal of the parallelogram $ABRC$. Let a be the point to which A is transferred. Let the angle $BAR = \alpha$, the angle $CAR = \beta$, the angle $RAa = \epsilon$. The perpendiculars BM, CN, to the diagonal AR, having been drawn, we shall have, by the known properties of the parallelogram,

$$AR = AM + AN, \text{ and } BM = CN;$$

that is,

$$V = S \cos. \alpha + S' \cos. \beta,$$
$$S \sin. \alpha = S' \sin. \beta.$$

Then, multiplying the former of these equations by cos. ϵ, the latter by sin. ϵ, and taking their sum, we shall have

$$V \cos. \epsilon = S \cos. (\alpha + \epsilon) + S' \cos. (\beta - \epsilon).$$

And, if from the point a there be drawn the normals ab, ac, ar, then,

$$\cos. \epsilon = \frac{Ar}{Aa};\ \cos. (\alpha + \epsilon) = \frac{Ab}{Aa};\ \cos. (\beta - \epsilon) = \frac{Ac}{Aa}.$$

Wherefore,

$$V . Ar = S . Ab + S' . Ac.$$

Thus, by summoning the momentums of two forces, we have the momentum of their resultant. And if the forces are more than two, by taking the sum of the resultant of the first two, and the momentum of the third, we shall, in like manner, have the momentum of the resultant of the three forces, which will consequently be equal to the sum of their momentums. And thus may we proceed for as many forces as there are, whatever be their number.

791. SCHOLIUM 3. If the space described by the point A in the direction of any of the forces, be not towards the same parts, to which the force itself tends, but toward the contrary parts, the momentum of that force must be taken negatively.

Thus, in the parallelogram of Fig. 71,

$$V . Ar = S . Ab - S' . Ac.$$

792. *Proposition II.* If several forces, acting upon a free point A, are in equilibrium, the sum of their momentums shall be equal to nothing: and *vice versâ.*

Since there is an equilibrium, the resultant $V=0$; wherefore (788) we shall have

$$0 = Sds + S'ds' + S''ds'' \ldots\ldots\ldots$$

Conversely, if this equation obtains, we shall have (788) $Vdu=0$: and since du cannot be equal to nothing, for any motion whatever of the point A, V must necessarily be equal to nothing. Wherefore, there will be an equilibrium.

793. SCHOLIUM 1. If the point A, urged by the forces S, S', $S'' \ldots\ldots$ against a resisting line or surface,

is in equilibrium upon it, let K be put for the pressure, which it exerts against that line or surface. This pressure will undoubtedly act in the direction of a straight line k perpendicular to the line or surface: for if it acted obliquely, it might be resolved into two, the one normal, the other tangential; and with this last it would not press, which is contrary to the hypothesis. Let now this resistance be supposed to be removed, and in its place to be substituted a force $-K$, acting in the direction of the straight line k. It is manifest that the equilibrium will subsist; and the point A being now altogether free, instead of the equation of the preceding article, we shall have this other,

$$0 = Sds + S'ds' + S''ds'' \ldots\ldots - Kdk.$$

794. Scholium 2. Further, in this case also, the equation of Art. 792. will obtain, upon the supposition that the point A moves upon the resisting line or surface, passing to the indefinitely near point a. For then the indefinitely small space Aa will be an infinitesimal straight line perpendicular to the straight line k; whence, the point A does not advance in the direction of k. Wherefore $dk = 0$, and the equation of Art. 793, becomes

$$0 = Sds + S'ds' + S''ds'' \ldots\ldots$$

795. *Proposition III.* If there be a system of points A, B, C, &c. acted upon by the forces S, S', S'', &c. and if this system be perfectly free, and in equilibrium, the sum of the momentums of the forces shall be equal to nothing.

For let the three points A, B, C, (Fig. 72.) be acted upon by the forces $AP = S$, $BQ = S'$, $CR = S''$. These

three points being connected by the straight lines AB, AC, BC, let $AB=f$, $AC=f'$, $BC=f''$. Let p denote the action which the point B exercises on the point A, in consequence of the connexion of the parts of the system; which action will take place in the direction AB, and will be equal and contrary to that which the point A exercises upon B: or, in other words, let p be the tension of the straight line AB. And, similarly, let p' be the tension of AC, and p'' the tension of BC.

It is clear that upon the point A there act the three forces S, p, p', which must be in equilibrium with one another, independently of the rest of the system: because the whole of the remainder of the system does not act upon A, except by means of the forces p, p'. Similarly, at the point B, the three forces S', p, p'', are in equilibrium with one another, and also the three forces S'', p', p'', at the point C.

Now there being given to the system an indefinitely small motion, such as the connexion of its parts allows, the point A will describe, in the direction of AP, the indefinitely small space ds, in the direction of AB, the indefinitely small space df, and in the direction of AC, the indefinitely small space df'. The point B will describe, in the directions BQ, BA, BC, the indefinitely small spaces ds', $d\phi$, df''. And, lastly, the point C will describe in the directions CR, CA, CB, the indefinitely small spaces ds'', $d\phi'$, $d\phi''$. We shall have (792) the three equations,

$$0= Sds + pdf + p'df',$$
$$0= S'ds' + pd\phi + p''df'',$$
$$0= S''ds'' + p'd\phi' + p''d\phi'';$$

summing which, we shall have, for the equilibrium of the whole system, the equation

$$0 = Sds + Sds' + S''ds'' + p(df + d\phi) + p'(df' + d\phi') + p''(df'' + d\phi'').$$

But the equilibrium of the system will equally subsist, if we suppose that the forces S, S', S'', are suppressed, and that the points A, B, C are kept fixt by the sole tensions p, p', p'', acting in a contrary direction. Wherefore the preceding equation will still subsist, if $S = S' = S'' = 0$, and if the sign of the forces p, p', p'', be changed. Therefore,

$$0 = -p(df + d\phi) - p'(df' + d\phi') - p''(df'' + d\phi''),$$

which equation added to the preceding, gives

$$0 = Sds + S'ds' + S''ds''.$$

It is easily understood how this demonstration may be extended to any number of points whatever, that compose the system.

796. Scholium 1. The demonstration is alike applicable to rigid systems and to those of a variable form. But for the former kind, we may reason also in the following manner. The rigid rod AB cannot change its place except by a motion compounded of a progressive and of a rotatory motion. Now, for any progressive motion whatever of the straight line AB, it is clear that the two extremities A, B, advance equally and in the same direction AB: and, for an indefinitely small rotatory motion, it is also clear, that, since both the points A, B, describe infinitesimal straight lines perpendicular to AB, neither of them advances in the direction of AB. Wherefore, when an indefinitely small motion has been given to the system,

the infinitesimal spaces described by the two points A, B, in the direction of the force p, must necessarily be equal and turned towards the same parts. Since, then, this force p acts upon the two points A, B, in opposite directions, one of these infinitesimal spaces will necessarily be described in the direction of the force p, and the other in a direction contrary to it. Therefore, in rigid systems, $d\phi = -df$. And similarly it will be found that $d\phi' = -df'$, and that $d\phi'' = -df''$. Whence the equation of the preceding article is easily reduced to this

$$0 = Sds + S'ds' + S''ds'' \ldots.$$

797. SCHOLIUM 2. If there be in the system any fixt point, let H be the pressure which it sustains, and let its direction be a straight line h. The equilibrium will subsist, even when this fulcrum is removed, if in its place we suppose to be applied to the system a force $-H$, in the direction of h. It will then be the same as if the system were free, and the following equation will obtain,

$$0 = Sds + S'ds' + S''ds'' \ldots . - Hdh.$$

798. SCHOLIUM 3. Similarly, if there be in the system any point propped by, or pressed against, a resisting surface, let k be the pressure sustained by the surface, directed according to the normal k. The equilibrium will not be disturbed, if this resistance be removed, and if, in its stead, the force $-K$ act upon the system in the direction k. Hence, the equation will become

$$0 = Sds + S'ds' + S''ds'' \ldots . - Kdk.$$

799. SCHOLIUM 4. Let us suppose, that the indefinitely small motion, given by our former hypothesis to

the system, of which the virtual velocities, and the momentums of the forces, are measured, is not any kind of motion, but such only as the obstacles that are in contact with the system, permit. In this case the fixt points must be understood to remain fixt; whence $h = 0$. And similarly the points pressed against a resisting line or surface must be understood to move upon it, describing an infinitesimal straight line, which will be perpendicular to k; whence $dk = 0$. Upon such an hypothesis, then, we have afresh the equation

$$0 = Sds + S'ds' + S''ds'' \dots,$$

as in free systems.

800. SCHOLIUM 5. Wherefore, in an equipoised system, the sum of the momentums of the forces will be found to be equal to nothing, for every indefinitely small motion, that is consistent with the constitution, and with the particular conditions of the system. And this sum will also be found equal to nothing, for any indefinitely small motion whatever, provided that amongst the forces applied to the system, the reactions, or the resistances of the obstacles in contact with it, are enumerated.

801. *Proposition IV.* If the sum of the momentums of the forces, applied to a system, is equal to nothing, the system shall be in equilibrium.

For let the system of points A, B, C, &c. be acted upon by the forces S, S', S'', &c. and let the sum of their momentums be equal to nothing. And if it be possible, let the system not be in equilibrium, and let the points A, B, C, &c. have communicated to them the velocities v, v', v'', &c. by which, in the instant dt, they describe

the small spaces $v\,dt$, $v'dt$, $v''dt$. Now, if to the points A, B, C, &c. already acted upon by the forces S, S', S'', &c. there be also applied the forces $-v, -v', -v''$, &c. which destroy in those points the initial velocities v, v', v'', &c. it is evident that, by the addition of these new forces, the system itself will be in equilibrium. Wherefore, the momentums of these new forces being v^2dt, v'^2dt, v''^2dt, &c. we shall have (795)

$$0 = Sds + S'ds' + S''ds'' \ldots .$$
$$- v^2dt - v'^2dt - v''^2dt.$$

But, by the hypothesis,

$$0 = Sds + S'ds' + S''ds'' \ldots . ;$$

wherefore, also,

$$0 = v^2dt + v'^2dt + v''^2dt \ldots . ,$$

which equation cannot hold, unless $v = 0$, $v' = 0$, $v'' = 0$. When, therefore, the sum of the momentums of the forces is equal to nothing, the points A, B, C, &c. cannot receive any motion whatever; and the system is therefore in equilibrium.

CHAP. II.

STATICAL THEOREMS DEDUCED FROM THE PRINCIPLE OF VIRTUAL VELOCITIES.

802. *Proposition I.* On a straight lever, two weights, that are in equilibrium with each other, are reciprocally proportional to their distances from the fulcrum.

This proposition, upon which rests the whole theory of machines, and which is therefore numbered amongst the principles of Statics, is immediately deducible from the principle of virtual velocities.

For, an indefinitely small rotation being supposed to be given to the lever AFB (Fig. 73.) about the fulcrum F, by which it passes to the position CFD, and the horizontal straight lines Ca, Db, terminated by the verticals Aa, Bb, having been drawn, it is plain that the virtual velocities Aa, Bb, of the two weights P, Q, hung at A and B, are proportional to the distances AF, BF. Now, by the principle of virtual velocities (795) we have $P.Aa - Q.Bb = 0$. Wherefore,

$$P : Q :: Bb : Aa :: BF : AF.$$

With equal facility it may be proved that, on the bent lever AFB (Fig. 74.) $Aa : Bb :: MF : NF$; and hence, when there is an equilibrium,

$$P : Q :: NF : MF.$$

803. *Proposition II.* If a system of heavy bodies be in equilibrium, and if any indefinitely small motion be given to the system, the centre of gravity of the system shall neither rise nor fall.

For let the weights of the bodies be S, S', S'', &c; let s, s', s'', &c. be vertical straight lines drawn from the centre of gravity of each of the bodies, and terminated by any horizontal plane: and let Z be the altitude of the centre of gravity of the system above that plane. Then (48)

$$Z = \frac{Ss + S's' + S''s'' \dots}{S + S' + S'' \dots}$$

Let now an indefinitely small motion be impressed upon the system, by which the verticals s, s', s'', &c. become $s + ds$, $s' + ds'$, $s'' + ds''$, &c. and Z becomes $Z + dZ$. Then,

$$dZ = \frac{S\,ds + S'ds' + S''ds'' \dots}{S + S' + S'' \dots}$$

But, by the hypothesis, the system is in equilibrium; wherefore, (795) $S\,ds + S'ds' + S''ds'' \dots = 0$; and, therefore, $dZ = 0$, and the altitude Z is constant.

804. *Coroll.* 1. Conversely, if a system of weights be so constituted as that, by any indefinitely small motion whatever, its centre of gravity neither rises nor falls, the system shall be in equilibrium. This theorem was propounded by Torricelli, who deduces from it the law of equilibrium of two heavy bodies upon an inclined plane; and from it, also, may be deduced the law of equilibrium in all the other machines, in which several weights counterbalance each other.

805. *Coroll.* 2. When $dZ = 0$, Z, generally speaking, will be either a maximum or a minimum. Whence it is inferred that, in a system of heavy bodies in equilibrium, the altitude of the centre of gravity above the horizon, is the greatest, or the least, of those which can obtain, if the system lose its situation of equilibrium, toward whatever part it may incline.

If this altitude is a minimum, the centre of gravity occupies the lowest place, and if the equilibrium be, in any manner, disturbed, the centre of gravity will rise. When the disturbing cause ceases to act, this centre will again begin to descend, and the system will settle itself in the situation of equilibrium. Thus it happens in a balance, in which the centre of gravity falls below the centre of motion.

If on the other hand, this altitude is a maximum, the centre of gravity is in its highest possible place, and, toward whatever side the system vibrates, the centre itself drops. When the disturbing cause ceases, the centre of gravity will still descend, and the system will continually depart further from the situation of equilibrium. Thus a balance falls over, in which the centre of gravity is posited above the centre of motion.

806. *Proposition III.* If a system of points be acted upon, in the directions of the straight lines s, s', s'', &c. by the forces S, S', S'', &c. such that the function

$$S ds + S' ds' + S'' ds'', \text{\&c.}$$

is an exact differential $= d\phi$, and if the system be in equilibrium, the function ϕ shall be a maximum, or a minimum: and *vice versâ*. This theorem, propounded

by Maupertius, is an immediate consequence of the principle of virtual velocities. Since, if there be an equilibrium, we have (795) $d\phi = 0$; whence ϕ is a maximum or a minimum: and *vice versâ*.

The inverse proposition is always and universally true: but the direct proposition is subject to some exception; it being a possible case that $d\phi = 0$, and that still the function ϕ is neither a maximum, nor a minimum.

807. Scholium. In nature, the acting forces are always such that the function $Sds + S'ds' + S''ds''$, &c. is an exact differential, and the preceding theorem obtains. For these forces are either constant and independent of the situation of the points to which they are applied, or they are attractions directed to a centre, and functions of the distances between that centre and the points upon which the force acts: so that taking these distances for the straight lines s, s', s'', &c. the forces S, S', S'', &c. will be either constant, or functions of s, s', s'', &c. respectively. In both cases it is manifest that the formula

$$Sds + S'ds' + S''ds'', \text{ \&c.}$$

becomes an exact differential $= d\phi$.

CHAP. III.

ON THE MODE OF DEDUCING FROM THE PRINCIPLE OF VIRTUAL VELOCITIES, THE CONDITIONS AND THE EQUATIONS OF EQUILIBRIUM.

808. Following the steps of Lagrange, we are now to explain the use of the principle of virtual velocities, in the resolution of Statical problems. And before we descend to particular cases, it may be well to indicate, generally, the track which we purpose to pursue.

Let then a system of points A, B, C, &c. be acted upon by the forces S, S', S'', &c. When this system is referred to three rectangular axes, let

$$x, y, z;\ x', y', z';\ x'', y'', z'', \text{\&c.}$$

be the co-ordinates of the points A, B, C, &c. Let the force S be resolved into three, P, Q, R, in the directions of x, y, z; and likewise the force S' into the three P', Q', R', &c. and so on. Making the sum of the momentums, (795.) equal to nothing, we shall have for the general equation of equilibrium.

$$\begin{aligned}(A)\quad 0 = {} & Pdx + P'dx' + P''dx'' \ldots \\ & + Qdy + Q'dy' + Q''dy'' \ldots \\ & + Rdz + R'dz' + R''dz'' \ldots\end{aligned}$$

809. If all the points A, B, C, of the system were free, and had no mutual connexion, so that every point could change its place independently of the rest, the variations of the co-ordinates dx, dy, dz, dx', &c. would be

all of them arbitrary and independent of one another, and the equation A would hold, whatever values were assigned to those variations. Hence it would be necessary to put their coefficients equal to nothing, and we should have the equations

$$P=0,\ Q=0,\ R=0,\ P'=0,\ \&c.$$

three times as many as there are points composing the system.

810. But if these points are retained, either by external obstacles, or by bars, which connect them with each other, or in any other manner, so that the motion of any one of these points having been determined at pleasure, the others are obliged to follow it, describing given lines, the variations dx, dy, dz, dx', &c. are then no longer arbitrary, but depend upon the particular conditions and constitution of the system. By means of these equations are eliminated, from the equation (A), so many of the variations dx, dy, &c. Those of them, which remain, will still be arbitrary and independent of one another. Their coefficients are, therefore, put equal to nothing; and thus are obtained the equations of equilibrium of the particular system proposed.

811. These equations may, also, be arrived at, by the following method.

The equilibrium of a system ought to subsist, even when the obstacles, which detain the points of the system, and the bars which connect them, are removed, provided that there are applied to those points forces equal and contrary to the actions, which the system exerts against those obstacles. Suppose, therefore, that to the points A, B, C, &c.

besides the forces S, S', S'', &c. there are applied the forces $-p$, $-q$, $-r$, &c., expressing the reactions of the obstacles, which retain the system; and let the equation (A) be also made to comprise the momentums of these forces. The case will then be the same, as if all the points of the system were free; all the variations dx, dy, dz, &c. become arbitrary; and their coefficients, being severally put equal to nothing, will furnish so many equations; from which, if we eliminate the quantities p, q, r, &c., the remaining equations will be those that express the equilibrium of the proposed system.

This latter method has an advantage over the former; inasmuch as, besides the conditions of equilibrium, it puts us in possession of the values of p, q, r, &c.; that is, if the actions exerted by the system against the obstacles, such as the pressures upon the fulcrums, the tensions of the bars, or of the strings, &c.

812. Of these two methods we may select, in the solution of particular problems, that which affords the greater facility. And it will be of great advantage to take care to express the conditions of the system in equations the best adapted to combine with the equation A, and to facilitate the elimination of the indeterminate quantities. This will be made more evident, by examples.

813. *Proposition I.* To determine the conditions of equilibrium of a free point.

The equation (A) in this case becomes

$$Pdx + Qdy + Rdz = 0,$$

whence, (809) we shall have

$$P=0,\ Q=0,\ R=0,$$

which result may be compared with Art. 95.

814. *Proposition II.* To determine the conditions of equilibrium of a point, which rests upon a resisting surface.

Let the equation of the surface be

$$l\,dx+m\,dy+n\,dz=0.$$

By means of this we may eliminate (810) from the equation

$$P\,dx+Q\,dy+R\,dz=0,$$

the indeterminate quantity dx; whence, there will remain

$$(Pm-Ql)\,dy+(Pn-Rl)\,dz=0.$$

$$\text{Therefore, } Pm-Ql=0;\ Pn-Rl=0;$$

$$\text{and } P : Q : R :: l : m : n,$$

as in Art. 102.

815. *Proposition III.* To determine the conditions of equilibrium of a rigid system.

Such a system cannot change its place, unless it be either by a progressive motion, or by a rotatory motion about one of the three rectangular axes, or by a motion compounded of these. Now, in the case of any progressive motion whatever, we shall have

$$dx=dx'=dx'',\text{ \&c.};\ dy=dy'=dy'',\text{ \&c};$$

$$dz=dz'=dz'',\text{ \&c.}$$

If we substitute these values in the equation (A), and make use of the symbol Σ briefly to denote the sum of the homologous quantities, belonging to different points of the system, so that, for example,

$$\Sigma'.P = P + P' + P'', \text{ \&c.}$$

that equation becomes

$$0 = dx\Sigma.P + dy.\Sigma Q + dz\Sigma.R,$$

whence we have these three conditions of equilibrium,

$$\Sigma.P = 0;\ \Sigma.Q = 0;\ \Sigma.R = 0.$$

If an indefinitely small rotation $d\phi$ be supposed to take place about OZ, (Fig. 75.) the axis of z, we shall have in the first place,

$$dz = dz' = dz'', \text{ \&c.} = 0.$$

Again, the similar triangles OPQ, Qrq, give

$$OQ : Qq :: PQ : qr :: OP : Qr;$$

$$\text{or, } 1 : d\phi :: y : dx :: x : -dy;$$

$$\text{whence, } dx = yd\phi;\ dy = -xd\phi.$$

And, similarly,

$$dx' = y'd\phi;\ dy' = -x'd\phi;$$

$$dx'' = y''d\phi;\ dy'' = -x''d\phi; \text{ \&c.}$$

Hence, the equation (A) immediately gives

$$0 = \Sigma.Py - \Sigma.Qx, \text{ or } \Sigma.Py = \Sigma.Qx.$$

In like manner, if an indefinitely small rotation be supposed to take place about OY, we shall have, for the fifth condition of equilibrium,

$$\Sigma.Pz = \Sigma.Rx;$$

and if such a rotation be supposed to take place about OX, we shall have for the sixth condition,

$$\Sigma.Qz = \Sigma.Ry.$$

816. SCHOLIUM 1. This is the shortest method of deducing, from the principle of virtual velocities, the conditions of equilibrium of a rigid system. We may also arrive at the same conclusion, in another way, by pursuing the track of Art. 810, but expressing differently the condition of the invariability of the system.

Let f, g, h, &c. denote the straight lines AB, AC, BC, &c. which join the points A, B, C, &c. of the system; and we shall have

$$f = \sqrt{\{(x'-x)^2 + (y'-y)^2 + (z'-z)^2\}},$$

$$g = \sqrt{\{(x''-x)^2 + (y''-y)^2 + (z''-z)^2\}},$$

$$h = \sqrt{\{(x''-x')^2 + (y''-y')^2 + (z''-z')^2\}}.$$

Now, on account of the invariable form of the system, these distances f, g, h, &c. must remain constant, however the system be put in motion. Hence, we have the equations

$$df = 0,\ dg = 0,\ dh = 0, \text{ \&c.},$$

by means of which we can eliminate, from the equation (A) so many of the indeterminate quantities

$$dx,\ dy,\ dz,\ dx', \text{ \&c.}$$

And making the coefficients, of those which remain, equal to nothing, there will result the same six equations as before.

817. SCHOLIUM 2. It may be seen, even without completing the computation, that, however many are the points composing the system, the final equations will always be six, neither more, nor fewer.

If these points be three, the equation (A) will involve nine indeterminate quantities; of which eliminating three, by the three equations $df=0$, $dg=0$, $dt=0$, there will remain six indeterminate arbitrary quantities, and as many equations.

But if the points be more than three, for every additional point, there will be introduced into the equation (A) three new indeterminate quantities; but there will also be an addition of three new equations, by which they may be eliminated. For the position of the new point is determined by its distances from the three first points. Let these distances be f', g', h', and we shall have the three new equations

$$df' = 0, \ dg' = 0, \ dh' = 0.$$

Thus, at the end of the eliminations, there will always remain in the equation (A) six indeterminate arbitrary quantities as at first.

818. Scholium 3. Lastly, we may also follow the method of Art. 811, expressing by p, q, r, &c. the tensions of the straight lines f, g, h, &c., and adding to the equation (A) the terms

$$-pdf, \ -qdg, \ -rdh, \text{ \&c.}$$

Then, considering all the indeterminate quantities as arbitrary, we may put the coefficient of each of them equal to nothing; and when the quantities p, q, r, &c., have been eliminated, we shall arrive at the same final equations, with the additional advantage of being enabled to determine the tensions themselves, p, q, r, &c.

819. *Proposition IV.* To determine the conditions of equilibrium of the funicular polygon, any forces what-

ever being supposed to be applied at the extremities of the sides.

These forces having been resolved into others parallel to three axes, we shall have (808) the equation (A). Here, the indeterminate quantities dx, dy, dz, dx', &c. will be three times as many as there are terminations of the sides of the polygon. Whence if the number of terminations be m, the number of indeterminate quantities will be $3m$.

Let now f, g, h, &c. be the sides of the polygon. Since the sides cannot be lengthened, we shall have the equations of condition $df=0$, $dg=0$, $dh=0$, &c. as many as there are sides of the polygon, that is to say $m-1$. By means of these, we can eliminate in equation (A) as many indeterminate quantities; and there will remain of them $2m+1$, arbitrary. If their coefficients be put equal to nothing, these will be the equations or conditions of equilibrium of the proposed polygon.

820. *Coroll.* 1. Or, putting p, q, r, &c. for the tensions of the ropes f, g, h, &c. there will be formed the equation

$$(A) \qquad -pdf-qdg-rdh\ldots\ldots=0;$$

and, proceeding according to the method indicated in Art. 818, we shall arrive at the same equations.

821. *Coroll.* 2. Here it is of use to point out, that we can always determine the position of all the terminations of the sides, and consequently, that of the whole polygon. For if to the equations which express the conditions of equilibrium, in number $2m+1$, we add the

equations of condition $df = 0$, &c. in number $m - 1$, we have $3m$ equations between the vertexes of the polygon; as many exactly as are sufficient to determine all the co-ordinates.

822. *Coroll.* 3. If the polygon be closed in, the number of terminations will be equal to the number of the sides; whence the arbitrary variations, and consequently the equations of equilibrium will be equal only to $2m$.

823. *Coroll.* 4. If the extreme ends of the polygon be fixt, there will be in the equation (A) six indeterminate quantities at the least; because there will be wanting the six variations belonging to the two first extremities; whence the equations of equilibrium will remain only $2m - 5$.

But if the extreme ends of the polygon are not fixt, but only forced to run upon given surfaces, the equations of these surfaces will serve to eliminate in the equation (A) two indeterminate quantities; so that the number of final equations will be restricted to $2m - 1$.

824. *Coroll.* 5. Ordinarily, it will be most convenient to seek first, the conditions of equilibrium of the polygon, considering its extreme ends as fixt; and to determine the tensions of the two last strings. Then, if the extreme euds of the polygon are not fixt, it will become further necessary that these tensions shall be in equilibrium with the forces, or the resistances, which are applied at the ends.

825. *Coroll.* 6. If any of the joints, at which the

forces are applied, be a running joint, that, for example, which unites the sides f, g, it is then no longer necessary that f and g be constant; it is sufficient if their sum $f+g$ be constant. Hence, instead of the two equations $df=0$, $dg=0$, we have the single equation $df+dg=0$. We have, therefore, one indeterminate quantity the less to eliminate; and we shall have, on that account, a condition of equilibrium the more.

826. SCHOLIUM. In the systems, which we have hitherto considered, the forces were applied to points, separate from each other. If we wish to subject a continuous system to the same method of reasoning, it will be proper to distinguish those variations of the co-ordinates, which arise when we pass from one point of the system to another, from those which arise when the first point changes its place in consequence of that indefinitely small motion of the system, from which we measure the virtual velocities of the forces, and their momentums. We shall denote the former variations by usual symbols dx, dy, dz; the latter, by the symbols δx, δy, δz. Let x, y, z, be the co-ordinates of any point M. Those of the next point will be $x+dx$, $y+dy$, $z+dz$; and those of the same point M, transferred to another position by the indefinitely small motion of the system, will be $x+\delta x$, $y+\delta y$, $z+\delta z$; and if the point M is acted upon by the forces P, Q, R, in the directions of the co-ordinates x, y, z, the momentums of these forces will be $P\delta x$, $Q\delta y$, $R\delta z$.

By the principle of virtual velocities, we must take the sum of the momentums of all the forces acting upon the system, and put it equal to nothing; and this will be the general equation of equilibrium. Now, if the extreme

points of the system are fixt, this equation may always be reduced to the following form :—

$$\int(L\delta x+M\delta y+N\delta z)=0,$$

where the symbol $\int$ denotes the integral or the sum of the momentums $L\delta x$, $M\delta y$, &c. taken for the whole extent of the proposed system.

For even when under the symbol $\int$ are included terms of the form $A\delta dx$, $A\delta ddx$, &c., these may always be transformed in the following manner. First, since the variations expressed by the symbols d, δ, are entirely independent of one another, these symbols may be changed at pleasure; so that instead of $A\delta dx$, $A\delta ddx$, &c. we may write $Ad\delta x$, $Add\delta x$, &c. Then, integrating partially, we have

$$\int A d\delta x = A\delta x - \int dA \,.\, \delta x;$$

and taking the integral for the whole extent of the system,

$$\int A\, d\delta x = A''\delta x'' - A'\delta x' - \int d\,A \,.\, \delta x,$$

where $A'\delta x'$ is the value acquired by $A\,\delta x$, in the first point of the system, and $A''dx''$ that acquired by it in the last point of the system. But these points are, by the hypothesis fixt; wherefore, $\delta x'=\delta x''=0$; whence

$$\int A\,d\delta x = -\int d\,A \,.\, \delta x.$$

And, in the same manner, it will be found that

$$\int A\,dd\delta x = \int dd\,A \,.\, \delta x.$$

By these transformations, we can always reduce the equation of equilibrium to the form above specified.

This being the case, if the elements of the system be all of them loose and unconnected, the variations δx, δy, δz, will be arbitrary and independent of one another; and

the equation of equilibrium will carry along with it (809) these three $L=0$, $M=0$, $N=0$. But if there be a connexion between the parts of the system, by which these variations become dependent upon one another, it will be proper to call in aid (810) the equations which express that relation; whence we may eliminate as many of these variations as is possible, and then put the coefficients, of those which remain, equal to nothing. Or, it may be convenient (811) to reckon amongst the forces applied to the elements of the system, the reactions also of the obstacles which impede them; and thus the variations δx, δy, δz, will become all of them independent of each other, and we may then proceed as in the first case.

But if the extreme points of the system are not fixt, it will be proper to consider also the terms $A'\delta x'$, &c. Commonly, however, it will be more convenient to investigate apart the conditions of equilibrium of those points, by the method indicated in Art. 824.

827. *Proposition* V. To find the curve of equilibrium of a flexible thread, acted upon in all its elements by given forces, the directions of which are all in the same plane.

By the hypothesis, the whole curve of the thread will lie in the same plane in which the forces act. Let Pds, Qds, be the forces which act upon the indefinitely small side ds, in the directions of x, y; and let T be the tension of that side. Considering the curve as a funicular polygon of an infinite number of sides, ds will represent any one of them as f; and, for the curve, the equation of the polygon (820) will become

$$\int(Pds\delta x + Qds\delta y - T\delta ds) = 0.$$

Now, in order to reduce this equation to the form proposed in Art. 826, it must be observed, that

$$\delta ds = \delta \sqrt{(dx^2 + dy^2)} = \frac{dx}{ds}\,\delta dx + \frac{dy}{ds}\,\delta dy.$$

Then (826) we have

$$\int \frac{T dx}{ds}\,\delta dx = -\int d\,.\,\frac{T dx}{ds}\,.\,\delta x,$$

$$\int \frac{T dy}{ds}\,\delta dy = -\int d\,.\,\frac{T dy}{ds}\,.\,\delta y.$$

By these substitutions the equation becomes

$$\int \left(P ds + d\,.\,\frac{T dx}{ds}\right) \delta x + \int \left(Q ds + d\,.\,\frac{T dy}{ds}\right) \delta y = 0.$$

Hence, equating with nothing the coefficients of the variations δx, δy, we have the equations

$$P ds + d\,.\,\frac{T dx}{ds} = 0; \quad Q ds + d\,.\,\frac{T dy}{ds} = 0;$$

whence, $-\dfrac{T dx}{ds} = \int P ds$; $-\dfrac{T dy}{ds} = \int Q ds$;

and eliminating T,

$$dy \int P ds - dx \int Q ds = 0,$$

the equation sought, which may be compared with Art. 165.

CHAP. IV.

ON THE APPLICATION OF THE PRINCIPLE OF VIRTUAL VELOCITIES TO BODIES IN MOTION.

828. D'ALEMBERT has taught, (299) in what manner the laws of the motion of a system may be deduced from those of its equilibrium. And as we have, from the principle of virtual velocities, arrived at an equation, which duly represents the equilibrium of a system acted upon by given forces, we may likewise thence deduce another equation proper for the determination of the motion, which will be impressed upon the system by given forces.

829. Let the points A, B, C, connected with each other in any given manner, be acted upon by given forces. Let P, Q, R, be the accelerating forces, which act upon the point A, in the directions of its co-ordinates x, y, z; and, at the end of the time t, let that point be moving with the velocity V, which is to be resolved into the three velocities u, v, w, in the directions of x, y, z, respectively. In the next instant, it will have the velocities

$$u + du, \quad v + dv, \quad w + dw.$$

But, by the free action of the accelerating forces, it ought (206) to have had the velocities

$$u + P dt, \quad v + Q dt, \quad w + R dt.$$

Wherefore, there will have been destroyed, by the opposition of the parts of the system, the velocities

$$P\,dt-du,\ Q\,dt-dv,\ R\,dt-dw.$$

If, in like manner, we distinguish by an accent, the homologous quantities belonging to the point B, we shall find that it loses the velocities

$$P'dt-du',\ Q'dt-dv',\ R'dt-dw';$$

and so of the other points. Now the forces, corresponding to these last velocities, ought (299) to be in equilibrium with one another. Wherefore, if we suppose the points A, B, C, &c. to be urged only by the forces $P\,dt-du$, &c. and if we imagine the system to receive such an indefinitely small motion, as is consistent with the particular conditions of the system, by which motion the co-ordinates x, y, z, of the point A, undergo the variations δx, δy, δz, and the co-ordinates x', y', z', of the point B undergo the variations $\delta x'$, $\delta y'$, $\delta z'$, and so on, we shall have, by the principle of virtual velocities, this equation

$$\begin{aligned}(B)\quad 0=&(P\,dt-du)\,\delta x+(P'dt-du')\,\delta x'\ldots\\&+(Q\,dt-dv)\,\delta y+(Q'dt-dv')\,\delta y'\ldots\\&+(R\,dt-dw)\,\delta z+(R'dt-dw')\,\delta z'\ldots\end{aligned}$$

830. Dividing by the element of time dt, which we shall consider as constant, and observing (206) that

$$u=\frac{dx}{dt},\ v=\frac{dy}{dt},\ w=\frac{dz}{dt},$$

the equation (B) may also be represented under this form,

$$(B)\ 0=\left(P-\frac{ddx}{dt^2}\right)\delta x+\left(P'-\frac{ddx'}{dt^2}\right)\delta x'\ldots$$
$$+\left(Q-\frac{ddy}{dt^2}\right)\delta y+\left(Q'-\frac{ddy'}{dt^2}\right)\delta y'\ldots$$
$$+\left(R-\frac{ddz}{dt^2}\right)\delta z+\left(R'-\frac{ddz'}{dt^2}\right)\delta z'\ldots$$

831. If now the points of the system be free from all connexion with each other, the variations δx, δy, $\delta x'$, &c. will be all arbitrary; we may, therefore, put their coefficients equal to nothing, and we shall have, for the determination of the motion of the point A, the three equations

$$Pdt=du;\quad Qdt=dv;\quad Rdt=dw;$$
$$\text{or } P=\frac{ddx}{dt^2};\quad Q=\frac{ddy}{dt^2};\quad R=\frac{ddz}{dt^2};$$

And for the motion of the point B, we shall have the three equations

$$P'dt=du';\ Q'dt=dv';\ R'dt=dw',\ \&c.$$

But if the tie which joins together the points of the system remain, it will be proper to make use of the equations of condition, in order to eliminate from the equation (B) as many as possible of the indeterminate quantities δx, δy, &c. Or else it will be proper to reckon, amongst the forces applied to the system, the reactions also of the obstacles. In a word, we must treat the equation (B) in the same manner as was taught with respect to the equation (A) in Arts. 809, 810, 811.

832. If the point A be the centre of gravity of a mass m, urged in all its elements by the accelerating forces P, Q, R, the forces acting upon the point A, will be

$$mP,\ mQ,\ mR;$$

and the forces destroyed will be

$$m\,(P\,d\,t - d\,u),\quad m\,(Q\,d\,t - d\,v),\quad m\,(R\,d\,t - d\,w).$$

And if, in like manner, in the points B, C, &c. there be the masses m', m'', &c. it is manifest, that in order to adopt the equation (B) to this system of masses, we must in the place of δx, δy, δz, put $m\,\delta x$, $m\,\delta y$, $m\,\delta z$; and in the place of $\delta x'$, $\delta y'$, $\delta z'$, we must put

$$m'\delta x',\quad m'\delta y',\quad m'\delta z',\ \&c.$$

CHAP. V.

DYNAMICAL THEOREMS DEDUCED FROM THE PRINCIPLE OF VIRTUAL VELOCITIES.

833. ***Proposition I.*** The centre of gravity of a free system moves in the same manner as if the whole mass of the system were there re-united, and as if all the forces were there immediately applied.

Let the system be composed of the masses m, m', m'', &c., considered as if they were so many weights. If the system be free, that is, unimpeded by fixt points, or other external obstacles, the centre of gravity will move as if it were one single mass $\Sigma . m$, equal to the mass of the whole system, which may be called M, urged by the forces

$$\Sigma . mP, \quad \Sigma . mQ, \quad \Sigma . mR.$$

For the equation (B) obtaining for every indefinitely small motion consistent with the conditions of the system, will also obtain for a progressive motion. Now, in the case of any progressive motion whatever, we have

$$\delta x = \delta x' = \delta x'', \text{ \&c.}; \; \delta y = \delta y' = \delta y'', \text{ \&c.};$$
$$\delta z = \delta z' = \delta z'', \text{ \&c.}$$

Substituting these values in the equation (B), and then putting the coefficients of the residual variations δx, δy, δz, equal to nothing, we shall have

$$\Sigma . mP = \Sigma . \frac{mddx}{dt^2};\ \Sigma . mQ = \Sigma . \frac{mddy}{dt^2};$$

$$\Sigma . mR = \Sigma . \frac{mddz}{dt^2}.$$

Let now X, Y, Z, be the co-ordinates of the centre of gravity, and we shall have

$$\Sigma . mx = MX;\ \Sigma . my = MY;\ \Sigma . mz = MZ;$$

whence

$$\Sigma . \frac{mddx}{dt^2} = \frac{MddX}{dt^2};\ \Sigma . \frac{mddy}{dt^2} = \frac{MddY}{dt^2};$$

$$\Sigma . \frac{mddz}{dt^2} = \frac{MddZ}{dt^2}.$$

Wherefore,

$$\frac{\Sigma . mP}{M} = \frac{ddX}{dt^2};\ \frac{\Sigma . mQ}{M} = \frac{ddY}{dt^2};\ \frac{\Sigma . mR}{M} = \frac{ddZ}{dt^2}.$$

But these equations are exactly (831) the equations that determine the motion of a point urged by the accelerating forces

$$\frac{\Sigma . mP}{M},\ \frac{\Sigma . mQ}{M},\ \frac{\Sigma . mR}{M};$$

or, which amounts to the same, of a mass M, urged by the forces

$$\Sigma . mP,\ \Sigma . mQ,\ \Sigma . mR.$$

834. *Coroll.* 1. If there be no accelerating forces and the system move solely in consequence of its acquired velocity, or of momentary impulses given to the different masses m, m', m'', &c., the centre of gravity will move equably in a straight line.

835. *Coroll.* 2. The same thing happens, if the accelerating forces consist of mutual attractions of one mass towards another, or of one point towards another.

For, since the attraction of the point A towards B is equal and contrary to that of the point B towards A, and so of all the other points, it is plain that, when these forces are transferred to the centre of gravity, the resultant will be nothing. Wherefore, by these attractions, the centre of gravity will not be put in motion at all; nor can it stir, except by the previously acquired velocities, or by momentaneous impulses given to the system; and this motion is (834) necessarily uniform and rectilinear.

836. *Coroll.* 3. Hence it may be inferred, that mutual action amongst the parts of a free system does not produce any change in the motion of its centre of gravity: and in this consists the principle known in Mechanics under the name of *Conservation of the Motion of the Centre of Gravity.*

837. Let a point move upon the curve BMS (Fig. 75.) describing, in the instant dt, the indefinitely small arc Mm. If from the centre O there be drawn to each of the points M, m, a radius vector, OM, and Om, the triangle MOm may be called the *Elementary Area* described, in the instant dt, about the centre O: and the sum of all the elementary areas constitutes the whole *Area* described in the time t.

Let Qy be the projection of the arc Mm, and OQ, Oq, the projections of OM, Om, upon any plane XOY: then, the triangle QOq will be the projection of the

elementary area; and the sum of all these projections constitutes the projection of the whole area described, in the time t, by the point M, about the centre O.

838. LEMMA I. The curve BMS (Fig. 75.) being referred to three rectangular axes OX, OY, OZ, drawn through the centre O, and the elementary area MOm being supposed to be projected upon the three planes XOY, XOZ, YOZ, if the three resulting projections be dA, dB, dC, then shall

$$2dA = ydx - xdy,$$
$$2dB = xdz - zdx,$$
$$2dC = zdy - ydz.$$

For it has been proved, in Art. 113, that if a line, resolved, in the manner as a force, in the directions of x, y, z, gives the three components P, Q, R, and if the triangle, formed by straight lines drawn from the origin of the axes to the extremities of that line, is projected upon the three planes XOY, XOZ, YOZ, there result the three projections

$$\tfrac{1}{2}(Py - Qx),\ \tfrac{1}{2}(Rx - Pz),\ \tfrac{1}{2}(Qz - Ry).$$

Applying this theorem to the line Mm, which, resolved in the directions of x, y, z, gives dx, dy, dz, and for P, Q, R, substituting respectively dx, dy, dz, the equations, above written, easily come out.

839. LEMMA 2. In every free system, acted upon by any forces whatever, the three following equations obtain:

$$\Sigma . m\frac{yddx - xddy}{dt^2} = \Sigma . m(Py - Qx);$$

$$\Sigma . m \frac{xddz - zddx}{dt^2} = \Sigma . m (Rx - Px);$$

$$\Sigma . m \frac{zddy - yddz}{dt^2} = \Sigma . m (Qz - Ry).$$

For since the system is free, it admits of a rotatory motion about any one of the axes. Now, an indefinitely small rotation $d\phi$ being supposed to take place about the axis of z, we shall have (815)

$$\delta x = y \delta \phi, \ \delta y = -x \delta \phi, \ \delta z = 0;$$
$$\delta x' = y' \delta \phi, \ \delta y' = -x' \delta \phi, \ \delta z' = 0, \text{ \&c.}$$

The equation (B), when these values are substituted in it, is transformed into the first of the announced equations. Similarly, if an indefinitely small rotation be supposed to take place about the axis of y, we shall have the second equation; and, such a rotation being supposed to take place about the axis of x, we shall have the third.

840. Scholium. When the system is free, these three equations are always and generally true, wherever the origin of the axes of the co-ordinates is placed, and however the axes are constituted.

If there be, in the system, a fixt point, the three equations will still obtain, provided that the origin of the co-ordinates is posited in that fixt point.

If there be two fixt points, or an immoveable axis, then taking this axis for that of z, for example, the first of the three equations will obtain, but not the other two.

841. *Proposition II.* If a free system be referred to three axes OX, OY, OZ; and if the accelerating forces P, Q, R, be such that their momentums of rotation about each of these three axes are nothing; and if the area described, in the time t by each mass m, about the origin O, be supposed to be projected upon the plane XOY; the sum of these projections multiplied each by the mass to which it belongs, shall be proportional to the time t: and the same shall be true of the projections made upon the planes XOZ, YOZ.

For since, by the hypothesis, the sum of the momentums of rotation of the forces P, Q, R, about each of the three axes, is nothing, we have (114)

$$\Sigma . m\,(Py - Qx) = 0,\ \Sigma . m\,(Rx - Pz) = 0,$$
$$\Sigma . m\,(Qz - Ry) = 0.$$

Again, calling, as above, A, B, C, the projections of the areas upon the planes XOY, XOZ, YOZ, we have (838)

$$y\,ddx - x\,ddy = d.(y\,dx - x\,dy) = 2\,ddA,$$
$$x\,ddz - z\,ddx = d.(x\,dz - z\,dx) = 2\,ddB,$$
$$z\,ddy - y\,ddz = d.(z\,dy - y\,dz) = 2\,ddC.$$

Hence, the three equations (839) become

$$\Sigma . \frac{2\,mddA}{dt^2} = 0;\ \ \Sigma . \frac{2\,mddB}{dt^2} = 0;\ \ \Sigma . \frac{2\,mddC}{dt^2} = 0.$$

Integrating, upon the supposition that dt is constant, and so that the integral may begin when $t = 0$, we have

$$\Sigma . 2\,mA = ct;\ \ \Sigma . 2\,mB = c't;\ \ \Sigma . 2\,mC = c''t;$$

c, c', c'' being constant coefficients. Wherefore, &c.

In this theorem consists the principle known by the name of the *Conservation of the Areas.*

842. *Coroll.* 1. The condition of the momentums of rotation of the accelerating forces, with respect to all the three axes, destroying each other, may be verified in many cases. First, when there are no accelerating forces, and the system moves by instantaneous impulses, or by previously acquired velocities, the condition is fulfilled, wheresoever the origin of the co-ordinates may have been placed, and whatever may have been the axes. For, in this case, if we consider the areas described about any point whatever, and their projections upon any plane that passes through that point, the sum of these projections multiplied each of them by the corresponding mass, must be proportional to the time.

843. *Coroll.* 2. In the second place, when the accelerating forces consist in mutual attractions between one point and another, since these forces, taken two and two, are equal and contrary, their momentums of rotation, in respect of any imaginable axis, destroy each other: and in this case, also, as in the former case, the theorem is true for areas described about any point whatever, and projected upon any plane, which passes through that point.

844. *Coroll.* 3. Thirdly, when the accelerating forces are all directed to one centre, the momentum of rotation of each of them will be nothing, with respect to every axis drawn through that centre. Hence, in this case, the theorem is true only for the areas described about the centre of the forces, and projected upon a plane which passes through that centre.

845. *Coroll.* 4. If a body is urged by an accelerating force directed to a centre, it will describe about it areas proportional to the times; and *vice versâ*.

Let O (Fig. 75.) be the centre of the force, and XOY a plane passing through O, and through the direction of the first velocity impressed upon the moveable body. It is plain that the trajectory will lie wholly in the plane XOY. Hence, the area described by the body is the same as A its projection. Now (844) $2mA = ct$; wherefore the area described is proportional to the time.

And, *vice versâ*, if $2mA = ct$, we shall have

$$\frac{2m\,ddA}{dt^2} = 0.$$

Wherefore (841) the momentum of rotation of the force, with respect to the axis OZ, will be nothing. Its direction will, therefore, cut the axis OZ; and, since it lies in the plane XOY, it will necessarily pass through the point O.

846. *Proposition III.* If a system be acted upon by forces, such that $Pdx + Qdy + Rdz$ is an exact differential $= d\phi$, and similarly

$$P'dx' + Q'dy' + R'dz' = d\phi', \text{ \&c.,}$$

then

$$\Sigma \,.\, mV^2 = C + 2\,\Sigma \,.\, m\phi,$$

C being a constant quantity.

For the equation (B) being true for every indefinitely small motion, which can take place in the proposed system, it will doubtless also be true for that indefinitely

small motion, which the system actually receives in the instant of time dt. We may, therefore, in the equation (B) make

$$\delta x = dx, \quad \delta y = dy, \quad \delta z = dz, \quad \delta x' = dx', \text{ \&c.:}$$

and, with these substitutions, it will become

$$\Sigma . m\,(udu + vdv + wdw) = \Sigma . m d\phi;$$

integrating, we have

$$\Sigma . \tfrac{1}{2} m\,(u^2 + v^2 + w^2) = \Sigma \tfrac{1}{2} m V^2 = \Sigma . m\phi + C.$$

Wherefore, &c.

847. *Coroll.* 1. If there are no accelerating forces, the sum of the forces, each of which is called (402) a *vis viva*, or the whole *vis viva* of the system, is constant: in which consists the principle of the *conservation of vires vivæ*.

848. *Coroll.* 2. But when there are accelerating forces, the *vis viva* changes, as the system passes from one situation to another; and increases or decreases accordingly as $\Sigma . m\phi$ increases or decreases.

In the beginning of motion let $V = U$, and $\phi = \Phi$. When, therefore, $t = 0$, we shall have

$$\Sigma . m U^2 = C + 2\,\Sigma . m\Phi.$$

But, at the end of the time t, we have

$$\Sigma . m V^2 = C + 2\,\Sigma m\phi.$$

Wherefore, the gain of *vis viva* will be

$$\Sigma . m V^2 - \Sigma . m U^2 = 2\,\Sigma . m\phi - 2\,\Sigma . m\Phi.$$

849. *Coroll.* 3. The value of ϕ depends solely upon the accelerating forces P, Q, R, and upon the co-ordi-

nates x, y, z, of the points to which they are applied. Whence it may be inferred, first, that if after a time t, the points A, B, C, &c. of the system shall have returned to their first situation, the *vis viva* will also have become what it was at first, whatever may have been the curves described by those points: secondly, that if these points have passed into other positions, as a, b, c, &c., the *vis viva* will indeed have suffered a change; but this change will remain the same, whatever may have been the curves described by the various points of the system.

850. *Coroll.* 4. Applying these reasonings to the motion of a single isolated body, it is clear, that, if this body is not urged by an accelerating force, its motion will be uniform, whatever be the curve which it describes.

For we shall have $\phi = 0$; and hence mV^2, and also V itself, will be constant.

851. *Coroll.* 5. But if it is urged by accelerating forces, and, setting out from the point A with a given velocity U, if it has reached the point a, the velocity, which it will have in a, will be the same, whatever has been the curve Aa, through which it has arrived there.

For (849) the change of *vis viva* $mV^2 - mU^2$, and consequently also the velocity V, is altogether independent of the curve Aa.

852. *Coroll.* 6. If there be a system of heavy bodies, which setting out from a state of rest, changes, in any manner, by motion, its situation, at every instant of the motion, the *vis viva* of the system will be the same as it

would have been, if each of the bodies had descended freely through an equal vertical altitude.

For, if the ordinates z be supposed to be vertical, $\phi = gz$, g being put for the gravity. If, therefore, at the beginning of the motion, $z = Z$, and $U = 0$, we shall have (848)

$$\Sigma . mV^2 = 2g\Sigma . mz - 2g\Sigma . mZ.$$

Suppose, now, each mass m to descend freely through the altitude $z - Z$; it will have acquired the velocity $\sqrt{2g(z-Z)}$; whence, its *vis viva* will be $2mg(z - Z)$, and the *vis viva* of the system will be as before

$$2g\Sigma . mz - 2g\Sigma . mZ.$$

853. *Coroll.* 7. According to the same hypothesis, the centre of gravity of the system will have descended as far, as it would ascend, if each body were to spring up vertically with the velocity acquired in the descent.

It is manifest that the descent of the centre of gravity is expressed by $\dfrac{\Sigma . mz - \Sigma . mZ}{\Sigma . m}$. Now if all the bodies of the system sprung up vertically with the velocity V, each of them (217) would rise through the vertical altitude $\dfrac{V^2}{2g}$; whence the rise of the centre of gravity would be $\dfrac{\Sigma . mV^2}{2g\Sigma . m}$. But we have seen (852) that

$$\Sigma . mV^2 = 2g\Sigma . mz - 2g\Sigma . mZ.$$

Wherefore, &c.

Huygens proposes this theorem, and avails himself of it in the investigation of the centre of oscillation: and

also, from this principle Daniel Bernouilli deduces the laws of Hydrodynamics.

854. *Coroll.* 8. Returning to a system urged by forces of any kind, let us suppose that, in the course of its motion, it passes through the situation, in which, if it had been placed at first, it would have remained in equilibrium: when the system has arrived at that position, its *vis viva* will there be a maximum, or a minimum; and *vice versâ*.

For, in the position of equilibrium, the sum of the momentums of the forces must (806) be equal to nothing, and therefore, $\Sigma . m\phi$ will be a maximum or a minimum; as, therefore, will also be $\Sigma . m V^2$.

Thus in an oscillating pendulum the *vis viva* is greatest, when the centre of gravity passes below the fulcrum; and least, when it passes over it. This theorem is due to Courtivron.

855. Scholium 1. All these theorems, on the conservation or the alteration of the *vis viva* of a system, which passes from one situation to another, are verified upon the supposition that, during this passage, no obstacle is encountered, that can suddenly change, by a finite quantity, the velocities of the bodies composing the system. For then, from one instant to another, the differences of the co-ordinates x, y, z, will be no longer infinitesimal, but finite. Nor can we then, as before (846) put the variations δx, δy, δz, equal to those differences: since, the principle of virtual velocities, generally speaking, obtains only for indefinitely small motions, and for the infinitesimal variations of the co-ordinates.

856. Scholium 2. Hence it is that, in the impact of hard bodies, there is a loss of *vis viva*; since the velocities change suddenly in the shock. If the velocity of each mass is resolved into two, the one, that which is preserved after the impact, the other, that which is destroyed in the impact, the portion of *vis viva* which corresponds to this latter velocity, is extinguished, whilst the other is preserved.

857. Scholium 3. But in the impact of elastic bodies, the velocity changing by indefinitely small degrees during the compression and the successive restitution of the parts, the law of the *vis viva* will obtain. If the elasticity be perfect, every particle will restore itself to the point whence it set out, and therefore (849) the *vis viva* again becomes what it was at first. If it is imperfect, the restitution not being completely effected, there will be a loss of *vis viva*.

858. *Proposition IV.* If a system be acted upon by forces such that each trinomial $Pdx+Qdy+Rdz$, &c. is an exact differential, and if, in the time t, the masses m, m', m'', &c. of the points A, B, C, &c., have passed to the points a, b, c, &c., through the curves Aa, Bb, Cc, &c.; then will these curves be such, that if each mass m is multiplied by the integral $\int V ds$, taken from the point of departure to the point of arrival a, the sum of these products is a minimum.

That is to say, $\Sigma . m \int V ds$ will have a less value than it would have, if the bodies described any other curves terminated by the same extremities A and a, B and b, &c.

Since (846)

$$\Sigma . m V^2 = C + 2 \Sigma . m (P dx + Q dy + R dz),$$

differentiating according to δ, we shall have

$$\Sigma . m V \delta V = \Sigma . m (P \delta x + Q \delta y + R \delta z).$$

Substituting this value in equation (B) of Art. 829, and observing that $V dt = ds$, we shall have

$$\Sigma . m ds \delta V = \Sigma . m (du \delta x + dv \delta y + dw \delta z).$$

Again, since $ds = \sqrt{(dx^2 + dy^2 + dz^2)}$, differentiating according to δ, and then dividing by dt, we shall find

$$V \delta ds = u \delta dx + v \delta dy + w \delta dz = u d \delta x + v d \delta y + w d \delta z,$$

and, therefore, also

$$\Sigma . m V \delta ds = \Sigma . m (u d \delta x + v d \delta y + w d \delta z).$$

Hence, we have

$$\begin{aligned} \Sigma . m \delta (V ds) &= \Sigma . (m ds \delta V + m V \delta ds) \\ &= \Sigma . md (u \delta x + v dy + w \delta z). \end{aligned}$$

And, integrating,

$$\begin{aligned} \Sigma . \int m \delta (V ds) &= \delta \Sigma . m \int V ds \\ &= C + \Sigma . m (u \delta x + v \delta y + w \delta z). \end{aligned}$$

Let, now, the integral be completed, so that, for each of the bodies, it may extend from the point of departure A to the point of arrival a; and it is manifest that, if we put M' for the value acquired by

$$\Sigma . m (u \delta x + v \delta y + w \delta z),$$

when the co-ordinates belong to the points of departure, and M'' for the value acquired by it, when they belong to the points of arrival, we shall have

$$\delta \Sigma . m \int V ds = M'' - M'.$$

But, by the hypothesis, the points of departure A, B, C, &c., and the points of arrival a, b, c, &c., are fixt and invariable, inasmuch as the curves comprised between these extremities are compared together. Wherefore, the variations of the co-ordinates belonging to those points, are equal to nothing. Therefore $M' = M'' = 0$; and $\delta\Sigma \,.\, m\,V ds = 0$; and, consequently, $\Sigma \,.\, m \int V ds$ is a minimum.

859. Scholium. There cannot be any thing equivocal, in our asserting that the preceding equation indicates a minimum, rather than a maximum: because $\int V ds$ may increase indefinitely the track which leads from the point A to the point a, being capable of an unlimited extension in length.

860. *Coroll.* 1. Applying the theorem to the motion of a single isolated body, we shall have $\delta \int V ds = 0$. Whence we gather this singular property of the trajectory described by a moveable point, that any two points whatever, A, a, being taken in it, the integral $\int V ds$, for the curve Aa, is less than it would be for any other curve joining the same extremities A, a.

861. *Coroll.* 2. If the body is not urged by accelerating forces, V will become constant, and $\delta s = 0$. Hence, the body will pass from the point A to the point a by the shortest possible line: and this will be evidently a straight line, if the point is free. But if the point moves upon a resisting surface, it will be the shortest line on that surface which joins the points A and a.

CHAP. VI.

ON THE MODE OF DEDUCING FROM THE PRINCIPLE OF VIRTUAL VELOCITIES THE EQUATIONS OF MOTION.

862. *Proposition I.* To determine the motion of a free point. The equation (B) furnishes at once these three

$$P = \frac{d\,d\,x}{d\,t^2},\; Q = \frac{d\,d\,y}{d\,t^2},\; R = \frac{d\,d\,z}{d\,t^2},$$

by which all the conditions of motion are determined.

This may be compared with Art. 231.

863. *Proposition II.* To determine the motion of a point, which is made to run over a given surface.

Let the equation of the given surface be

$$l\,d\,x + m\,d\,y + n\,d\,z = 0,$$

an indefinitely small motion being supposed to be given to the point upon the surface, by which its co-ordinates x, y, z, acquire the increments δx, δy, δz, we shall necessarily have

$$l\,\delta x + m\,\delta y + n\,\delta z = 0.$$

By means of this equation let δx be eliminated from equation (B), and let the coefficients of the other two variations δy, δz, which remain undetermined, be put equal to nothing. We shall then have the two equations

$$Pm - Ql = \frac{mddx - lddy}{dt^2}; \quad Pn - Rl = \frac{nddx - lddz}{dt^2},$$

to which adding the equation of the surface

$$ldx + mdy + ndz = 0,$$

we have given three equations, whence may be found for any time whatever t, the three co-ordinates x, y, z, that indicate the place of the moveable point.

The two equations above found are the very same as those which result from eliminating K, from the three equations of Art. 252.

864. *Proposition III.* To determine the motion of a free solid body, of invariable figure.

The three equations of Art. 833, and the three of Art. 839, all of them derived from the fundamental equation (B), include the full and complete determination of the motion. Calling M the mass of the body, dM the element of the mass, and, instead of the symbol Σ, making use of the sign of integration, since we are treating of a continued system, the three equations of Art. 833, become

$$\int P dM = \int dM \cdot \frac{ddx}{dt^2},$$

$$\int Q dM = \int dM \cdot \frac{ddy}{dt^2},$$

$$\int R dM = \int dM \cdot \frac{ddz}{dt^2}.$$

And the three equations of Art. 839, become

$$\int \frac{dM}{dt}(yddx - xddy) = \int dMdt\,(Py - Qx),$$

$$\int \frac{dM}{dt}(xddz - zddx) = \int dMdt\,(Rx - Pz),$$

$$\int \frac{dM}{dt}(zddy - yddz) = \int dMdt\,(Qz - Ry).$$

These three equations are the same as those which, by another method, are found in Art. 380; and it is there shewn, that by them the motion is completely determined.

865. Scholium. If the body is sustained by a fulcrum, or an immoveable pin, the three first equations no longer obtain; the three last subsist, provided that (840) the origin of the co-ordinates is placed in the fixt point; and by them is determined the rotation of the body about its prop.

But if the body is tied to an immoveable axis, it will be proper to take this for one of the axes of the co-ordinates, for that of z, for example. Then will the first only of the three latter equations obtain, and by it will the rotation of the body be determined.

866. *Proposition IV.* To determine the motion of a flexible cord, fixt at one of its extremities, acted upon in every one of its points by given forces, and in any manner moved from the rectilineal position, in which it would be in equilibrium. Let the origin of the abscissas be taken in the fixt extremity of the cord, x in the rectilineal direction of the cord, when it is in equilibrium, and let the ordinates y be supposed to be perpendicular to x. Then, ds being any element whatever of the bent cord,

let P, Q, be the forces acting upon it in the directions of the co-ordinates x, y; let h be the thickness or transverse section of the cord, in this element, and let q be its density; whence, $h\,q\,d\,s$ will be the mass of the element $d\,s$. Lastly, let T be the tension of the cord, in the point determined by the co-ordinates x, y. For this system, the equation (B) will be

$$0 = \int \left(P - \frac{d\,d\,x}{d\,t^2}\right) h\,q\,d\,s\,\delta\,x$$

$$+ \int \left(Q - \frac{d\,d\,y}{d\,t^2}\right) h\,q\,d\,s\,\delta\,y - \int T\,\delta\,d\,s.$$

The last term, by the reductions taught in Art. 826, and practised in Art. 827, is converted into these two

$$\int \delta\,x\,d\,.\,\frac{T\,d\,x}{d\,s} + \int \delta\,y\,d\,.\,\frac{T\,d\,y}{d\,s}.$$

Substituting this expression, and then putting the coefficients of $\delta\,x$, $\delta\,y$ equal to nothing, we shall have, for the determination of the motion of the cord, these two equations

$$\frac{d\,d\,x}{d\,t^2} = P + \frac{1}{h\,q\,d\,s}\,d\,.\,\frac{T\,d\,x}{d\,s}\,;\quad \frac{d\,d\,y}{d\,t^2} = Q + \frac{1}{h\,q\,d\,s}\,d\,.\,\frac{T\,d\,y}{d\,s}.$$

867. Scholium. Before we apply the equations above found, to any particular case, it will be proper to remark, that the differences $d\,d\,x$, $d\,d\,y$, in the first members, are relative only to the variability of the time, and express only the variation of the co-ordinates of one and the same point in different times; as is manifest (829) from their generation. On the contrary, in the second members, the difference $d\,s$, and $d\,x$, $d\,y$ derived from it, (827) suppose

the time to be constant, and express only the variation of the elements x, y, s, through different points of the curve, and in the same time. It will, therefore, be convenient to distinguish them by the usual symbols of partial differences.

868. *Coroll.* 1. If we suppose the cord to depart but very little from the rectilineal position, we may put

$$\left(\frac{ddx}{dt^2}\right)=0, \text{ and } ds=dx.$$

Hence, the first equation will give

$$T=\theta-\int Phqds,$$

θ being a constant quantity. And the second, this value having been substituted, and the second member being differentiated upon the supposition that ds is constant, will become

$$\left(\frac{ddy}{dt^2}\right)-\left(\frac{\theta-\int Phqds}{hq}\right)\cdot\left(\frac{ddy}{dx^2}\right)$$
$$+P\left(\frac{dy}{dx}\right)-Q=0.$$

869. *Coroll.* 2. If a chain, of uniform weight and thickness, hang freely from above, and if it be moved very little from the vertical, we must put $P=g$, $Q=0$; and we shall have $T=\theta-ghqs$. Let l be the whole length of the chain; and it is manifest that when $s=l$, then $T=0$; whence

$$\theta=ghql, \text{ and } T=ghq(l-s).$$

Wherefore $\left(\frac{ddy}{dt^2}\right)-g(l-s)\cdot\left(\frac{ddy}{dx^2}\right)+g\left(\frac{dy}{dx}\right)=0.$

This equation exhibits the law of the oscillations of the chain; but it cannot be integrated by any known methods.

870. If an elastic cord, of uniform thickness and density, be fixt at both its extremities, and stretched by a weight θ; and if also it be void of gravity, or rather if its weight be neglected, as being very small in comparison of the stretching weight; in this case, making $P = Q = 0$, we shall have $T = \theta$, and the equation

$$\left(\frac{d\,d\,y}{d\,t^2}\right) - \frac{\theta}{h\,q}\left(\frac{d\,d\,y}{d\,x^2}\right) = 0;$$

of which the complete integral is

$$y = F\left(x + t\sqrt{\frac{\theta}{h\,q}}\right) + f\left(x - t\sqrt{\frac{\theta}{h\,q}}\right),$$

where the symbols F, f, denote arbitrary functions.

871. *Coroll.* 4. These functions are determined, whenever the initial figure of the cord is known, and also the initial velocity impressed on each of its points. These things being supposed to be known, we shall know, for every abscissa x, the value of y, and of $-\left(\frac{dy}{dt}\right)$, when $t = 0$. If, now, for the sake of abbreviation, we make $\sqrt{\frac{\theta}{hq}} = b$, we have, generally,

$$y = F(x + b\,t) + f(x - b\,t)$$

$$-\left(\frac{dy}{dt}\right) = -b\,F'(x + b\,t) + b\,f'(x - b\,t),$$

denoting, by the symbols F', f', the differentials of F, f;

whence, $d\,F(x + b\,t) = (d\,x + b\,d\,t)\ F'(x + b\,t)$.

Wherefore, if for any abscissa x the initial ordinate be s, and the initial velocity u, we shall have

$$s = F.x + f.x;\quad u = -b\,F'.x + b\,f'.x;$$

and multiplying the last equation by dx, and then integrating it, we have

$$-\int u\,dx = b\,F.x - bf.x. \text{ Wherefore,}$$

$$F.x = \tfrac{1}{2}S - \frac{1}{2b}\int u\,dx,$$

$$f.x = \tfrac{1}{2}s + \frac{1}{2b}\int u\,dx;$$

where the second members will be known functions of x. If in the first equation, for x we put $x+bt$, and if, in the second, for x we put $x-bt$, we shall thus have expressed, in x and t, the values of the two functions

$$F(x+bt),\quad f(x-bt),$$

the sum of which gives the value of y.

872. *Coroll.* 5. But here it is proper to notice that we have the values of s and of u, and thence of $F.x$ and of $f.x$, only for the abscissas x, comprised between the limits 0 and l, l being put for the length of the cord, from one end to the other. Consequently, the method above taught cannot give us the functions

$$F(x+bt),\quad f(x-bt),$$

except when $x \pm bt$ happens to be comprised between these limits. But the time t increases continually; and very little is enough to make $x+bt$ become greater than l, and $x-bt$ negative; whence we can no longer, in the continuation of the time, determine the motion and the figure of the cord. But another given condition supplies this defect; which is, that the two extremities of the cord being fixt, y must be $=0$, when $x=0$ and when $x=l$, whatever t may be; whence we have

$$F.bt + f.-bt = 0;\quad F(l+bt) + f(l-bt) = 0,$$

where t may be any positive number.

Wherefore, in general,

$$F.z = -f.-z; \quad F(l+z) = -f(l-z),$$

z being any positive number whatever; for which if we put successively $l \pm z$, $2l \pm z$, $3l \pm z$, &c. we shall have from the first equation

$$F(\ l \pm z) = -f(-\ l \mp z),$$
$$F(2l \pm z) = -f(-2l \mp z),$$
$$F(3l \pm z) = -f(-3l \mp z),$$
$$F(4l \pm z) = -f(-4l \mp z),$$
$$\text{\&c.} = \text{\&c.}$$

And, from the second equation,

$$F(2l \pm z) = -f. \mp z,$$
$$F(3l \pm z) = -f(-l \mp z),$$
$$F(4l \pm z) = -f(-2l \mp z),$$
$$\text{\&c.} = \text{\&c.};$$

whence are deduced the following conclusions,

$$F.z = F(2l+z) = F(4l+z), \text{ \&c.}$$
$$= -f(-2l-z) = -f(-4l-z), \text{ \&c.},$$
$$f.z = f(-2l+z) = f(-4l+z), \text{ \&c.}$$
$$= -F(2l-z) = -F(4l-z), \text{ \&c.}$$

From the first conclusion it is readily inferred, that whenever we know the value of $F.z$, for every number z comprised between 0 and l, we can know it also for any positive number whatever, however much it may exceed l. And, similarly, from the second conclusion it appears that, when the value of $f.z$ is given for every number z com-

prised between 0 and l, the value of $f.z$ for every negative number, is thence deducible. Thus we have what was wanting to enable us to complete the calculation of the values of y, indicated in the preceding article.

873. *Coroll.* 6. In the equation

$$y = F(x + b\,t) + f(x - b\,t).$$

If there be put successively $b\,t = 2\,l,\ 4\,l,\ 6\,l$, &c. the first term will successively be the values

$$F(2l + x),\ F(4l + x),\ \&c.,$$

each of them (872) equal to $F.x$; and the second term will take the values

$$f(-2l + x),\ f(-4l + x),\ \&c.$$

each of them equal to $f.x$. Wherefore there will remain always $y = F.x + f.x$, as it was at the beginning of motion.

Hence it is evident that the vibrating cord returns periodically to its initial position, at equal intervals of time $= \dfrac{2l}{b}$.

874. *Coroll.* 7. If, in the same equation, we put successively $bt = l,\ 3l,\ 5l$, &c., and if, in the same time, the origin of the abscissas be transferred to the other extremity of the cord, so that x is changed into $l - x$, then the first term $F(x + b\,t)$, having become $F(l - x + b\,t)$, will take successively the values

$$F(2l - x),\quad F(4l - x),\ \&c.$$

each (872) equal to $-fx$. And the other term $f(x - bt)$, having become $f(l - x - b\,t)$, will acquire the values

$$f.-x,\ f(-x - 2l),\ \&c.$$

each equal to $-F.x$. Wherefore, we shall always have

$$y=-F.x-f.x;$$

which is the initial value, but with a contrary sign.

Hence it appears, that after the time $t=\frac{l}{b}$ has elapsed from the beginning of the motion, the cord will again resume the same curvature, as that which it had at first, but doubly reversed; that is to say, from below to above, and from right to left; and it afterwards returns to this position periodically, at intervals of time $=\frac{2l}{b}$.

875. *Coroll.* 8. The cord oscillates, therefore, after the manner of a pendulum, completing each oscillation in the time $t=\frac{l}{b}=\frac{l\sqrt{hq}}{\sqrt{\theta}}$. Its vibrations are isochronal, and performed in equal times with those of a simple pendulum of the length $\frac{ghql^2}{\pi^2\theta}$. And the number of vibrations, made in a given time, is in a ratio composed of the sub-duplicate ratio of the tension, of the inverse ratio of the length, and of the inverse sub-duplicate ratio of the thickness and of the density.

876. *Coroll.* 9. If no initial motion be impressed upou the cord, we shall have $u=0$; hence,

$$f.x=F.x, \text{ and } y=F(x+bt)+F(x-bt).$$

If the cord, at the beginning be stretched in a right line, we shall have $s=0$; whence

$$f.x=-F.x, \text{ and } y=F(x+bt)-F(x-bt).$$

In both cases, there is but one single function to be determined, and the computation becomes more simple.

Whosoever is desirous of seeing the geometrical construction of the vibrating cord explained, with minute particularity, and with wonderful clearness, may consult a most beautiful dissertation by Euler, in the third volume of the *Miscellanea Taurinensia.*

I.

A TABLE

OF NUMBERS WHICH ARE OF FREQUENT USE IN THE CALCULATIONS OF MECHANICS.

A.

Of accelerating forces that which most frequently occurs is gravity, denoted by g; and to it as to a standard measure, the rest are referred.

Now 1″ being the unity of time, and the *metre* the unity of space, we have (221.) in mean latitudes,

$$g = 9.8087952,$$
$$\log. g = 0.9916157,$$
$$\text{compl. log. } g = 9.0083843.$$

$$2g = 19.6175904,$$
$$\log. 2g = 1.2926457,$$
$$\text{compl. log. } 2g = 8.7073543.$$

If the Paris *foot* be taken for the unity of space,

$$g = 30.1957875,$$
$$\log. g = 1.4799465,$$
$$\text{compl. log. } g = 8.5200535.$$

$$2g = 60.3915750,$$
$$\log. 2g = 1.7809765,$$
$$\text{compl. log. } 2g = 8.2190235.$$

And, taking for unity the Paris *inch*,

$$g = 362.3494499,$$
$$\log. g = 2.5591277,$$
$$\text{compl. log. } g = 7.4408723.$$

$$2g = 724.6988998,$$
$$\log. 2g = 2.8601577,$$
$$\text{compl. log. } 2g = 7.1398423.$$

But, if we take the English *foot* for the unity of space,

$$g = 32.1816762,$$
$$\log. g = 1.5076086,$$
$$\text{compl. log. } g = 8.4923914.$$

$$2g = 64.3633523,$$
$$\log. 2g = 1.8086387,$$
$$\text{compl. log. } 2g = 8.1913613.$$

And if the English *inch* be taken for the unity of space,

$$g = 386.1801141,$$
$$\log. g = 2.5867899,$$
$$\text{compl. log. } g = 7.4132101.$$

$$2g = 772.3602281,$$
$$\log. 2g = 2.8878199,$$
$$\text{compl. log. } 2g = 7.1121801.$$

These latter values are set down, in order to facilitate a comparison of experiments and computations, which are found reported, in different treatises of Mechanics, in different denominations.

(B.)

In calculating the motions of bodies, which make isochronal vibrations, it is usual to seek the length of the *simple pendulum*, which performs its oscillations in the

same time as the vibrating body: and this length is commonly compared with that of the pendulum, which vibrates *seconds*. Thus, the time of each vibration (285) is easily found, and the number of vibrations made in a given time. Now, if the length of the pendulum, which vibrates seconds, be called l, we have (421) in mean latitudes, taking the *metre* for unity,

$$l = 0.9938387,$$
$$\log. l = 9.9973160,$$
$$\text{compl. log. } l = 0.0026840.$$

In Paris *feet*, we should have

$$l = 3.0594340,$$
$$\log. l = 0.4856273,$$
$$\text{compl. log. } l = 9.5143727.$$

In English *feet*, we should have

$$l = 3.2606854,$$
$$\log. l = 0.5133088,$$
$$\text{compl. log. } l = 9.4866912.$$

(C.)

If π denote the ratio of the circumference of a circle to its diameter, or that of the semi-circumference to the radius, we have

$$\pi = 3.1415926,$$
$$\log. \pi = 0.4971499,$$
$$\text{compl. log. } \pi = 9.5028501.$$

$$2\pi = 6.2831852,$$
$$\log. 2\pi = 0.7981799,$$
$$\text{compl. log. } 2\pi = 9.2018201.$$

II.

A TABLE

OF VELOCITIES AND OF THE ALTITUDES DUE TO THEM, EXPRESSED IN METRES.

A.

Velocities.	Altitudes.*	Velocities.	Altitudes.
0.10	0.0005	0.34	0.0059
0.11	0.0006	0.35	0.0062
0.12	0.0007	0.36	0.0066
0.13	0.0009	0.37	0.0070
0.14	0.0010	0.38	0.0074
0.15	0.0011	0.39	0.0078
0.16	0.0013	0.40	0.0082
0.17	0.0015	0.41	0.0086
0.18	0.0016	0.42	0.0090
0.19	0.0018	0.43	0.0094
0.20	0.0020	0.44	0.0099
0.21	0.0022	0.45	0.0103
0.22	0.0025	0.46	0.0108
0.23	0.0027	0.47	0.0113
0.24	0.0029	0.48	0.0117
0.25	0.0032	0.49	0.0122
0.26	0.0034	0.50	0.0127
0.27	0.0037	0.51	0.0133
0.28	0.0040	0.52	0.0138
0.29	0.0043	0.53	0.0143
0.30	0.0046	0.54	0.0149
0.31	0.0049	0.55	0.0154
0.32	0.0052	0.56	0.0160
0.33	0.0055	0.57	0.0166

Velocities.	Altitudes.	Velocities.	Altitudes.
0.58	0.0172	0.91	0.0422
0.59	0.0177	0.92	0.0432
0.60	0.0184	0.93	0.0441
0.61	0.0190	0.94	0.0451
0.62	0.0196	0.95	0.0460
0.63	0.0202	0.96	0.0470
0.64	0.0209	0.97	0.0480
0.65	0.0215	0.98	0.0490
0.66	0.0222	0.99	0.0500
0.67	0.0229	1.00	0.0510
0.68	0.0236	1.01	0.0520
0.69	0.0243	1.02	0.0530
0.70	0.0250	1.03	0.0541
0.71	0.0257	1.04	0.0551
0.72	0.0264	1.05	0.0562
0.73	0.0272	1.06	0.0573
0.74	0.0279	1.07	0.0584
0.75	0.0287	1.08	0.0595
0.76	0.0295	1.09	0.0606
0.77	0.0302	1.10	0.0617
0.78	0.0310	1.11	0.0628
0.79	0.0318	1.12	0.0639
0.80	0.0326	1.13	0.0651
0.81	0.0335	1.14	0.0662
0.82	0.0343	1.15	0.0674
0.83	0.0351	1.16	0.0686
0.84	0.0360	1.17	0.0698
0.85	0.0368	1.18	0.0710
0.86	0.0377	1.19	0.0722
0.87	0.0386	1.20	0.0734
0.88	0.0395	1.21	0.0746
0.89	0.0404	1.22	0.0759
0.90	0.0413	1.23	0.0771

Velocities.	Altitudes.	Velocities.	Altitudes.
1.24	0.0784	1.57	0.1256
1.25	0.0796	1.58	0.1272
1.26	0.0809	1.59	0.1289
1.27	0.0822	1.60	0.1305
1.28	0.0835	1.61	0.1321
1.29	0.0848	1.62	0.1338
1.30	0.0862	1.63	0.1354
1.31	0.0875	1.64	0.1371
1.32	0.0888	1.65	0.1388
1.33	0.0902	1.66	0.1405
1.34	0.0915	1.67	0.1422
1.35	0.0929	1.68	0.1439
1.36	0.0943	1.69	0.1456
1.37	0.0957	1.70	0.1473
1.38	0.0971	1.71	0.1490
1.39	0.0985	1.72	0.1508
1.40	0.0999	1.73	0.1526
1.41	0.1014	1.74	0.1543
1.42	0.1028	1.75	0.1561
1.43	0.1043	1.76	0.1579
1.44	0.1057	1.77	0.1597
1.45	0.1072	1.78	0.1615
1.46	0.1087	1.79	0.1633
1.47	0.1102	1.80	0.1652
1.48	0.1117	1.81	0.1670
1.49	0.1132	1.82	0.1688
1.50	0.1147	1.83	0.1707
1.51	0.1162	1.14	0.1726
1.52	0.1178	1.85	0.1745
1.53	0.1193	1.86	0.1763
1.54	0.1209	1.87	0.1782
1.55	0.1225	1.88	0.1802
1.56	0.1240	1.89	0.1821

Velocities.	Altitudes.	Velocities.	Altitudes.
1.90	0.1840	2.23	0.2535
1.91	0.1860	2.24	0.2558
1.92	0.1880	2.25	0.2581
1.93	0.1899	2.26	0.2604
1.94	0.1919	2.27	0.2627
1.95	0.1938	2.28	0.2650
1.96	0.1958	2.29	0.2673
1.97	0.1978	2.30	0.2697
1.98	0.1998	2.31	0.2720
1.99	0.2019	2.32	0.2744
2.00	0.2039	2.33	0.2767
2.01	0.2059	2.34	0.2791
2.02	0.2080	2.35	0.2815
2.03	0.2101	2.36	0.2839
2.04	0.2121	2.37	0.2863
2.05	0.2142	2.38	0.2887
2.06	0.2163	2.39	0.2912
2.07	0.2184	2.40	0.2936
2.08	0.2205	2.41	0.2961
2.09	0.2227	2.42	0.2985
2.10	0.2248	2.43	0.3010
2.11	0.2269	2.44	0.3035
2.12	0.2291	2.45	0.3060
2.13	0.2313	2.46	0.3085
2.14	0.2334	2.47	0.3110
2.15	0.2356	2.48	0.3135
2.16	0.2378	2.49	0.3160
2.17	0.2400	2.50	0.3186
2.18	0,2422	2.51	0.3211
2.19	0.2445	2.52	0.3237
2.20	0.2467	2.53	0.3263
2.21	0.2490	2.54	0.3289
2.22	0.2512	2.55	0.3315

Velocities.	Altitudes.	Velocities.	Altitudes.
2.56	0.3341	2.79	0.3968
2.57	0.3367	2.80	0.3996
2.58	0.3393	2.81	0.4025
2.59	0.3419	2.82	0.4054
2.60	0.3446	2.83	0,4083
2.61	0.3472	2.84	0.4111
2.62	0.3499	2.85	0.4140
2.63	0.3526	2.86	0.4169
2.64	0.3553	2.87	0.4199
2.65	0.3580	2.88	0.4228
2.66	0.3607	2.89	0.4257
2.67	0.3634	2.90	0.4287
2.68	0.3661	2.91	0.4317
2.69	0.3689	2.92	0.4346
2.70	0.3716	2.93	0.4376
2.71	0.3744	2.94	0.4406
2.72	0.3771	2.95	0.4436
2.73	0.3799	2.96	0.4466
2.74	0.3827	2.97	0.4496
2.75	0.3855	2.98	0.4527
2.76	0.3883	2.99	0.4557
2.77	0.3911	3.00	0.4588
2.78	0.3939		

B.

Altitudes.	Velocities.	Altitudes.	Velocities.
0.005	0.3132	0.160	1.7717
0.010	0.4429	0.165	1.7992
0.015	0.5425	0.170	1.8262
0.020	0.6264	0.175	1.8529
0.025	0.7003	0.180	1.8791
0.030	0.7672	0.185	1.9051
0.035	0.8286	0.190	1.9306
0.040	0.8858	0.195	1.9559
0.045	0.9396	0.200	1.9808
0.050	0.9904	0.205	2.0054
0.055	1.0387	0.210	2.0296
0.060	1.0849	0.215	2.0537
0.065	1.1292	0.220	2.0774
0.070	1.1718	0.225	2.1009
0.075	1.2130	0.230	2.1241
0.080	1.2527	0.235	2.1471
0.085	1.2913	0.240	2.1698
0.090	1.3288	0.245	2.1923
0.095	1.3652	0.250	2.2146
0.100	1.4006	0.255	2.2366
0.105	1.4352	0.260	2.2584
0.110	1.4690	0.265	2.2801
0.115	1.5020	0.270	2.3014
0.120	1.5343	0.275	2.3227
0.125	1.5641	0.280	2.3437
0.130	1.5970	0.285	2.3645
0.135	1.6274	0.290	2.3852
0.140	1.6572	0.295	2.4056
0.145	1.6866	0.300	2.4260
0.150	1.7154	0.305	2.4461
0.155	1.7488	0.310	2.4660

Altitudes.	Velocities.	Altitudes.	Velocities.
0.315	2.4858	0.410	2.8360
0.320	2.5055	0.415	2.8533
0.325	2.5250	0.420	2.8704
0.330	2.5444	0.425	2.8875
0.335	2.5636	0.430	2.9044
0.340	2.5826	0.435	2.9212
0.345	2.6015	0.440	2.9380
0.350	2.6203	0.445	2.9546
0.355	2.6390	0.450	2.9712
0.360	2.6575	0.455	2.9876
0.365	2.6759	0.460	3.0040
0.370	2.6942	0.465	3.0203
0.375	2.7123	0.470	3.0365
0.380	2.7303	0.475	3.0526
0.385	2.7482	0.480	3.0686
0.390	2.7660	0.485	3.0846
0.395	2.7837	0.490	3.1004
0.400	2.8013	0.495	3.1162
0.405	2.8187	0.500	3.1319

III.

A TABLE

OF THE LINEAR MEASURES OF DIFFERENT COUNTRIES, EXPRESSED IN METRES; AND VICE-VERSA.

	Foot expressed in Metres.	Metre expressed in Feet.
Ancona	0.4096	2.4416
Bergamo	0.4378	2.2843
Bologna	0.3801	2.6309
Brescia	0.4710	2.1232
Como	0.4512	2.2162
Cremona	0.4835	2.0681
Florence	0.5830	1.7152
Fermo	0.4245	2.3559
Ferrara	0.4039	2.4761
Forli	0.4882	2.0483
Genoa	0.2491	4.0145
London	0.3048	3.2809
Macerata	0.5585	1.7905
Mantua	0.4669	2.1420
Milan *Braccio*	0.5949	1.6808
Milan *Foot*	0.4352	2.2979
Modena	0.5230	1.9119
Naples	0.2620	3.8166
Novara	0.4709	2.1234
Padua	0.3574	2.7980
Paris	0.3248	3.0784
Rhenish Foot	0.3138	3.1869
Reggio	0.5309	1.8836
Rome *Palmo*	0.2234	4.4762
Rome *Ancient Foot*	0.2953	3.3865
Turin *Foot*	0.5137	1.9467
Treviso	0.4081	2.4503
Udine	0.3405	2.9369
Venice	0.3477	2.8758
Verona	0.3429	2.9162
Vienna	0.3161	3.1635

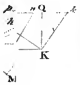
IV.

A TABLE

OF THE WEIGHTS OF DIFFERENT COUNTRIES, EXPRESSED IN CHILOGRAMS, OR METRIC POUNDS; AND VICE-VERSA.

		Pounds expressed in Chilograms.	Chilograms expressed in Pounds.
Amsterdam		0.4830	2.0703
Ancona		0.3296	3.0341
Bergamo		0.3251	3.0757
Bologna		0.3619	2.7636
Brescia		0.3208	3.1171
Como		0.3667	3.1579
Cremona		0.3095	3.2311
Fermo		0.3210	3.1155
Ferrara		0.3451	2.8974
Florence and Rome		0.3393	3.9469
Forli		0.3294	3.0354
Genoa		0.3168	3.1568
London	*Pound* Troy = 12 oz.	0.3731	2.6800
	Pound Av. = 16 oz.	0.4536	2.2045
Macerata; *as at Florence and Rome.*		......	......
Mantua		0.3105	3.2203
Milan		0.3268	3.0600
Modena		0.3405	2.9372
Naples		0.3208	3.1176
Novara		0.3255	3.0724
Padua	*Libbra sottile*	0.3389	2.9509
	Libbra grossa	0.4865	2.0553
Paris		0.4895	2.0429
Reggio		0.3245	3.0814
Turin		0.3688	2.7112
Treviso	*Libbra sottile as at Padua.*	......	......
	Libbra grossa	0.5167	1.9352
Venice	*Libbra sottile*	0.3012	3.3197
	Libbra grossa	0.4770	2.0964
Verona	*Libbra sottile*	0.3332	3.0014
	Libbra grossa	0.4998	2.0009
Vienna		0.5600	1.7857

THE END.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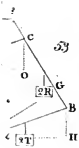
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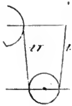

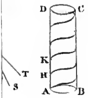

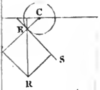

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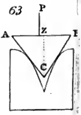

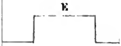
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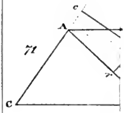
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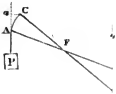
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